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HOMOGENIZED MODELS WITH MEMORY EFFECT FOR HETEROGENEOUS PERIODIC MEDIA

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Abstract. The homogenization of initial boundary value problems for heat conduction equations with asymptotically degenerate rapidly oscillating periodic coefficients are considered. Such problems model thermal processes in heterogeneous periodic media. Homogenized problems (whose solutions determine approximate asymptotics for solutions of the original problems) are presented. Estimates for the accuracy of the asymptotics and relevant convergence theorem are discussed. The homogenized problems have the form of initial boundary value problems for integro-differential equations in convolutions. The presence of convolutions in models for media is called the memory effect. Statements about the solvability and regularity for the problems and the homogenized problems are proved. These results are optimal even in the case of zero convolutions, when the homogenized problems coincide with the classical heat conduction problems.

Key words: Heat conduction equations, approximate asymptotics, solvability result.

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1. Introduction

The purpose of this article is to investigate models of thermal processes in a composite material with a periodic structure, which occupies a bounded domain Ω and is composed of two materials with very different properties. It is assumed that the domain Ω is represented as a union of two domains Ω_1^ε and Ω_0^ε with a periodic structure determined by a small positive parameter ε, which are separated by a common boundary $\partial\Omega_1^\varepsilon \setminus \partial\Omega$. The density and conductivity coefficients of the material corresponding to Ω_1^ε are constant, while the density coefficient of another material is characterized by the positive bounded parameter μ and the conductivity is equal to ε^2. The latter equality is usually called the case of *double porosity* in accordance with [2], where this case was first considered without a proof of the relevant convergence theorem and an accuracy estimate.

The study of the models will be based on homogenization methods. Initially, these methods were developed by I. Babuska, N. S. Bakhvalov, A. Bensoussan, J.-L. Lions, and G. Papanicolau to simplify computer simulations for composite material models (references can be found, for example, in [3, 5]). Accordance to

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the method the simulation of thermal processes in a composite with a periodic structure defined by ε can be replaced by the simulation of a thermal process for a homogeneous (homogenized) material with guaranteed accuracy for sufficiently small ε. In a mathematical sense, this means that the solution of the heat conduction problem with complex periodic coefficients is guaranteed to be close to the solution of the problem with constant coefficients. Appropriate theorems and accuracy estimates were proved, for example, in [3,5] for sufficiently regular data.

It also turned out that it is much easier to prove that the solution of some problem with complex periodic coefficients converges in an appropriate space to the solution of a problem with constant coefficients. Such an approach does not require usually additional data regularity but does not provide a guaranteed approximation accuracy, further details can be found, for example, in [5, 19]. Problems with complex periodic coefficients have two natural (micro-macro) scales and relevant fast and slow variables, which are reflected through rapidly oscillating coefficients. Therefore, later it turned out that it is even easier to prove the two-scale convergence of solutions of the problems to the solution of some homogenized two-scale problems. This approach is used, for example, in [10, 13, 14, 17].

However, such two-scale problems depend on two fast and slow variables, and the type of relevant equations for the problems is not clear. In addition, the accuracy of the approximations in this case has not been proven. Numerical methods for such two-scale (micro-macro) models are discussed, for example, in [11,12,15]. Further details on the approaches and relevant references can be found in [10, 12, 14].

When studying composite materials composed of materials with very different properties, additional small and large parameters may appear in models for such composites. Models for periodic composites already have one small parameter and new parameters can and should be correlated with the scale parameter ε, this can affect the homogenized problem for such composites.

Homogenization for some non-stationary initial boundary value problems with several parameters is considered in [21–24], where homogenized problems are obtained and accuracy estimates are proved. Under certain geometric assumptions on the composite structure, such homogenized problems are multiphase models, which are described by coupled systems of equations with convolutions [22–24]. The presence of convolutions in models for media is called *the memory effect* [9].

To obtain such homogenized models with convolutions in [21–24], the Laplace transform was used to transform non-stationary problems into stationary problems with a parameter, to which the asymptotic methods from [20] were applied. After obtaining the terms of the asymptotics, accuracy estimates are proved by energy methods. In the resulting problems, the fast and slow variables are separated, and the homogenized models with convolutions depend only on slow variables. The same approach is also used, for example, in articles on related topics [6, 7, 28, 29]. Homogenization of models in which there is initially a memory effect and which is preserved for the homogenized models is considered in [4, 18], where statements on the convergence and front structures are proved.

The results of [21–24] were obtained under the assumption that the initial data are regular enough and the initial conditions are homogeneous. Here we consider the general case of irregular data without homogeneity conditions. The precise statement of the problems is formulated in the next section, where the homogenized problems, solvability assertions, and accuracy estimates are given. As a consequence of this, convergence theorems for general data will be presented. The proofs of the assertions are given in the concluding section. To investigate the solvability of the homogenized problems with memory, we use the Laplace transform method developed in [1] to study parabolic problems. In the case $\mu = 1$ the results presented here are partially announced in [25, 26].

2. Initial boundary value problems for thermal processes

For an integer $n \geq 2$, assume that there are given a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and functions $f \in L^2(0,\infty; H^{-1}(\Omega))$ and $w \in L^2(\Omega)$. Here and below, function spaces are used, which are defined, for example, in [9]. We define a function $u = u(t,x)$ as a solution to the following problem

$$\begin{aligned} &m^\mu_\varepsilon\, u'_t \,-\, \operatorname{div}(\lambda^\varepsilon \nabla u) \,=\, r^\mu_\varepsilon f \quad \text{ in } \quad \Omega \times (0,\infty), \\ &u\big|_{t=0} = w \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega \times (0,\infty), \end{aligned} \tag{2.1}$$

which depends on a finite parameter μ (in what follows, $0 < \mu \leq 1$ for definiteness) and the small positive parameter ε in the following way.

Let E_1 be an open connected 1-periodic subset of $\mathbb{R}^n$ with locally Lipschitz boundary, $E_0 = \mathbb{R}^n \setminus \overline{F_1}$ be a subset with locally Lipschitz boundary and

$$E^\varepsilon_1 = \varepsilon E_1 = \{\varepsilon x : x \in E_1\}, \quad E^\varepsilon_0 = \varepsilon E_0 = \{\varepsilon x : x \in E_0\}.$$

Here and below, the 1-periodicity of a set (or a function) means that characteristic function of the set (or the function itself) is 1-periodic with respect to each of the independent variables x_i, $i = 1,\dots,n$. Thus, $Y \equiv (0,1)^n$ is a periodicity cell and the sets E_1 and E_0 with the common boundary ∂E_1 are completely determined by the sets $Y_1 = E_1 \cap Y$ and $Y_0 = E_0 \cap Y$ with the boundary $\Gamma = \partial E_1 \cap Y$. The sets Y_1 and Y_0 split the periodicity cell Y into two sets corresponding to different materials of the composite under consideration that are separated by a common boundary Γ. Examples illustrating such partitions are shown in Fig. 2.1.

The sets E^ε_0 and E^ε_1 for fixed ε define the periodic media $\Omega^\varepsilon_0 = E^\varepsilon_0 \cap \Omega$ and $\Omega^\varepsilon_1 = E^\varepsilon_1 \cap \Omega$, which are bounded by the boundary $\partial\Omega$ of Ω, where problem (2.1) is considered. Schematic drawings illustrating the media are shown in Fig. 2.2.

For the so-defined models of periodic media Ω^ε_0 and Ω^ε_1, corresponding to two different materials in the domain Ω, the dependence of the real coefficients of problem (2.1) on the parameter ε and μ is given by the following equalities

$$\lambda^\varepsilon = \varepsilon^2 \lambda_0 \quad \text{in} \quad \Omega^\varepsilon_0 \qquad \text{and} \qquad \lambda^\varepsilon = \lambda_1 \quad \text{in} \quad \Omega^\varepsilon_1,$$
$$m^\mu_\varepsilon = \mu\, m_0,\ \ r^\mu_\varepsilon = \mu\, r_0 \quad \text{in} \quad \Omega^\varepsilon_0 \quad \text{and} \quad m^\mu_\varepsilon = m_1,\ \ r^\mu_\varepsilon = r_1 \quad \text{in} \quad \Omega^\varepsilon_1,$$

where m_0, r_0, m_1 and r_1 are constants and the constant matrices λ_0 and λ_1 are symmetric and elliptic. Here the ellipticity means the existence of constants α and β such that the inequalities

$$0 < \alpha E \le \lambda_0 \le \beta E, \qquad 0 < \alpha E \le \lambda_1 \le \beta E$$

are true in matrix sense, where E is an identity matrix. Suppose also that

$$\alpha \le m_0 \le \beta, \quad \alpha \le m_1 \le \beta, \quad r_0 \le \beta, \quad r_1 \le \beta.$$

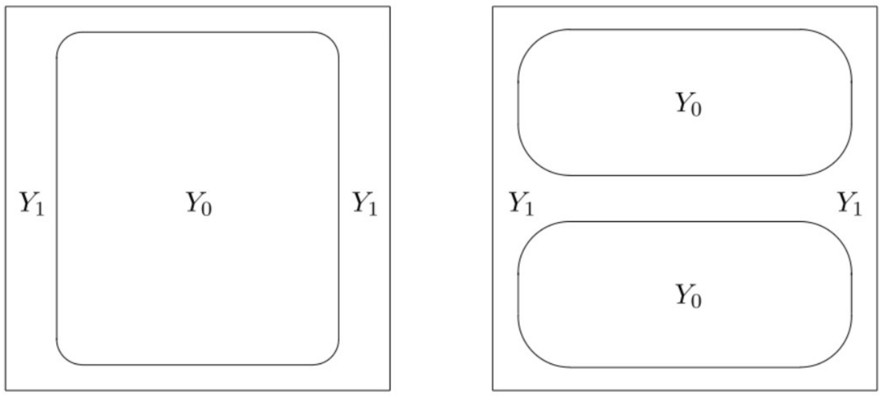

Fig. 2.1. The model of splitting a cell for different materials

We assumed that the sets Y_1 and Y_0 have positive Lebesgue measures in $\mathbb{R}^n$ and therefore the sets are not empty. Thus, for small ε, the equation of problem (2.1) is degenerated on the set Ω_0^ε, which simulates a weakly conductive material.

This dependence on the parameters leads to the homogenized problem with convolutions, the solutions of which approximate the solution of problem (2.1) for small ε uniformly in μ according to [24]. For a precise formulation of such homogenized initial-boundary value problems, additional definitions are needed.

Let the vector function $N = N(y)$ be a 1-periodic solution of the following Neumann problems on Y_1:

$$-\operatorname{div}_y(\lambda_1 \nabla_y N) = 0 \quad \text{in} \quad Y_1, \quad -(\lambda_1 \nabla_y N, \Upsilon) = (\lambda_1, \Upsilon) \quad \text{on} \quad \Gamma,$$

where Υ denotes the outward normal to the boundary $\Gamma = \partial Y_1$. Further, we introduce the following notation for the matrix with constant components

$$\Lambda = |Y_1|^{-1} \int_{Y_1} (\lambda_1 + \lambda_1 \nabla_y N(y))\, dy,$$

where $|Y_1|$ denotes the Lebesgue measure of the set Y_1. It is known [3,5] that the solution $N(y)$ and the matrix are well defined and Λ is symmetric and elliptic.

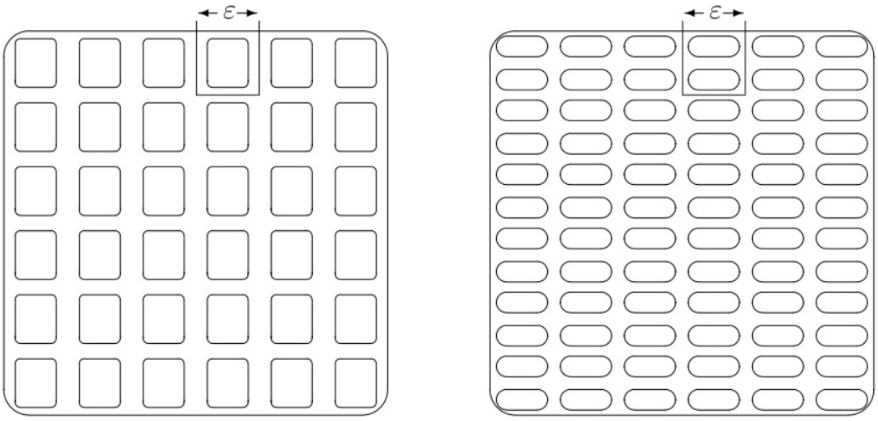

Fig. 2.2. Composite material models

In addition, let the functions $q = q(t,y)$ and $g = g(t,y)$ be 1-periodic solutions of the initial boundary value problems on Y_0:

$$\begin{array}{l} \mu\, m_0\, q_t' - \mathrm{div}_y(\lambda_0 \nabla_y\, q) = 0 \quad \text{in} \quad Y_0 \times (0,\infty), \\ q\big|_{t=0} = 1 \quad \text{in} \quad Y_0, \quad q = 0 \quad \text{on} \quad \overline{Y}_1 \times (0,\infty), \\ \mu\, m_0\, g_t' - \mathrm{div}_y(\lambda_0 \nabla_y\, g) = 0 \quad \text{in} \quad Y_0 \times (0,\infty), \\ g\big|_{t=0} = r_0/m_0 \quad \text{in} \quad Y_0, \quad g = 0 \quad \text{on} \quad \overline{Y}_1 \times (0,\infty). \end{array} \tag{2.2}$$

It is known [9] that suitable solutions to the problems exist and the functions

$$M(t) \;=\; |Y_1|^{-1}\!\int_{Y_0} m_0\, q_t'(t,y)\, dy, \quad R(t) \;=\; |Y_1|^{-1}\!\int_{Y_0} m_0\, g_t'(t,y)\, dy \tag{2.3}$$

are defined as elements of the space $L^1(0,\infty)$ in accordance with [23, 24].

The homogenized convolution problem for the function $v = v(t,x)$ is

$$\begin{array}{l} m\, v_t' - \mu\, M * (v_t') - \mathrm{div}(\Lambda \nabla v) = r\, f - \mu\, R * f \quad \text{in} \quad \Omega \times (0,\infty), \\ v\big|_{t=0} = w \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{array} \tag{2.4}$$

where $m = m_1$, $r = r_1$, and $*$ denotes the convolution operator by t, for example,

$$M * (v_t') \;=\; \int_0^t M\,(t-\tau)\,(v_\tau'(\tau,x))\, d\tau.$$

For fixed ε and μ, a unique solution to problem (2.1) exists, for example, according to [9]. For sufficiently smooth data and $w = 0$, a unique solution to problem (2.4) exists in accordance with [24]. Moreover, the solutions of problems (2.2) and (2.4) approximate the solution of problem (2.1) in the appropriate sense for small ε uniformly in μ. More precisely, it was proved in [24], that the following statement is fulfilled for the solutions to these problems.

Theorem 2.1. *Assume that $f \in C_0^\infty((0,T)\times\Omega)$ and $w = 0$. Let u be a solution to problem* (2.1)*, v be a solution to problem* (2.4)*, and T be fixed. Then*

$$\|u - v\|^2_{C^0([0,T];L^2(\Omega_1^\varepsilon))} + \mu \left\|u - v + \, q_\varepsilon * (v'_t) - g_\varepsilon * f\right\|^2_{C^0([0,T];L^2(\Omega_0^\varepsilon))} \leq C\,\varepsilon,$$

where $q_\varepsilon = q(t, x/\varepsilon)$ and $g_\varepsilon = g(t, x/\varepsilon)$ are defined by using solutions to problems (2.2) *and the constant C does not depend on parameters ε and μ for $0 < \varepsilon \leq \varepsilon_0$ with an appropriate positive ε_0 and $0 < \mu \leq 1$.*

Remark 2.1. In this theorem, a part of the energy norm for problem (2.1) is estimated. This part depends on μ, which is natural, since the density coefficient in problem (2.1) depends on μ on the part of the domain Ω. Using the methods of [20] and [24] to construct further terms in the asymptotics, one can estimate the entire energy norm in such estimate by the small parameter ε. However, the exact formulation of the relevant statement is much more cumbersome.

Thus, instead of solving problem (2.1), it is possible to solve problem (2.4) with some guaranteed accuracy. Naturally, the numerical solution of problem (2.1) for very small ε is much more complicated than the numerical solution of problem (2.4), since for the media shown in Fig. 2.2, a very fine mesh is required, taking into account the geometry of very small components of the composite material.

Moreover, according to the estimate, the solution to the original problem (2.1) is strongly oscillating on small components, which should also be displayed in the numerical solution. It is these oscillations that lead to the appearance of convolutions in the homogenized problem, which is also commonly called the *appearance of memory* in composite materials. Thus, the presence of weakly conductive components in the domain Ω is modeled by the appearance of a memory in density (coefficient at the time derivative) in homogenized composites.

Remark 2.2. The estimate of Theorem 2.1 is uniform with respect to the parameter μ. Related to this is the dependence of problems (2.2) and (2.4) on the parameter. This dependence can be eliminated by representing the solutions of problems (2.2) in the form $q = \tilde{q}(t/\mu, y)$ and $g = \tilde{g}(t/\mu, y)$. With this choice, the dependence on μ in (2.4) will also be illuminated, since the coefficients M and R are determined in terms of derivatives. Nevertheless, for small μ these functions turn out to be insignificant in a certain sense. In this connection, the following simple assertion will be proved in the next section.

Lemma 2.1. *Let q and g be solutions to problem* (2.2)*. Then*

$$\|q\|^2_{L^2(0,\infty;L^2(Y))} + \|g\|^2_{L^2(0,\infty;L^2(Y))} \leq C\mu,$$

where the constant C does not depend on μ for $0 < \mu \leq \mu_0$ with an appropriate μ_0.

For the small parameter μ, as a corollary of this assertion and results [24], the following statement will be proved.

Theorem 2.2. *Assume that $f \in C_0^\infty((0,T)\times\Omega)$ and $w=0$. Let u be a solution to problem* (2.1)*, v_0 be a solution to problem* (2.4) *with $\mu=0$, and T be fixed. Then*

$$\|u-v_0\|^2_{C^0([0,T];L^2(\Omega_1^\varepsilon))} + \mu\,\|u-v_0\|^2_{C^0([0,T];L^2(\Omega_0^\varepsilon))} \le C\left(\varepsilon+\mu^2\right),$$

where the constant C does not depend on parameters ε and μ for $0<\varepsilon\le\varepsilon_0$ and $0<\mu\le\mu_0$ with appropriate positive ε_0 and μ_0.

Thus, for low density at small μ and low conductivity for the material Ω_0^ε, memory effects do not arise for composite materials within the accepted accuracy. Nevertheless, for an arbitrary finite μ it is natural to prove the solvability and regularity for problem (2.4) with common initial data, since it is necessary for the numerical solution of this problem with guaranteed accuracy.

As a result, the following statement will be proved in the next section.

Theorem 2.3. *Let $f\in L^2(0,\infty;H^{-1}(\Omega))$ and $w\in L^2(\Omega)$. Then, the unique solution $v\in L^2(0,\infty;H_0^1(\Omega))$ to problem* (2.4) *exists and there is a positive constant C depending only on α, β, $|Y_0|$, and $|Y_1|$, such that*

$$\|v\|_{L^2(0,\infty;H_0^1(\Omega))} + \|v_t'\|_{L^2(0,\infty;H^{-1}(\Omega))} \le C\left(\|f\|_{L^2(0,\infty;H^{-1}(\Omega))} + \|w\|_{L^2(\Omega)}\right)$$

and $v\in C^0([0,T];L^2(\Omega))$ for fixed positive T.

Moreover, $v_t'\in L^2(0,\infty;L^2(\Omega))$ if $f\in L^2(0,\infty;L^2(\Omega))$ and $w\in H_0^1(\Omega)$.

The theorem proof will use the Laplace transform, which is an isomorphism in the appropriate sense [1]. After applying the Laplace transform to the equation from (2.4), the convolution with $-\mu M$ turns into multiplication by the function $-\mu\hat{M}$, which is defined on the upper complex half-plane. It will be verified that such a function is non-negative and bounded on this half-plane. Thus, in the Laplace image, the density for the homogenized equation from (2.4) is determined in terms of m plus a non-negative bounded function and, as it were, is spread over the entire half-plane. In contrast, under the conditions of Theorem 2.2 the homogenized density is the constant m for sufficiently small μ.

As a consequence of Theorems 2.1 and 2.3, we obtain the following statement on the convergence of solutions to problem (2.1) for common initial data.

Corollary 2.1. *Assume that $f\in L^2(0,T;L^2(\Omega))$, $w=0$, and T is fixed. Let u be a solution to problem* (2.1) *and v be a solution to problem* (2.4)*. Then*

$$\|u-v\|^2_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))} + \mu\left\|u-v+\,q_\varepsilon*(v_t')-g_\varepsilon*f\right\|^2_{L^\infty(0,T;L^1(\Omega_0^\varepsilon))} \to 0$$

for $\varepsilon\to 0$, where q_ε and g_ε are defined as in Theorem 2.1 and $0<\mu\le 1$.

Passing to the antiderivative of the solution of problem (2.1), we can obtain a simple convergence statement for the case of general $w\in L^2(\Omega)$. Similarly, we can deduce from Theorem 2.2 the following convergence statement.

Corollary 2.2. *Assume that* $f \in L^2(0,T;L^2(\Omega))$, $w=0$, *and* T *is fixed. Let* u *be a solution to problem* (2.1) *and* v_0 *be a solution to problem* (2.4) *with* $\mu=0$. *Then*

$$\|u - v_0\|^2_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))} + \mu\, \|u - v_0\|^2_{L^\infty(0,T;L^2(\Omega_0^\varepsilon))} \to 0$$

for $\varepsilon \to 0$ *and* $\mu \to 0$.

In conclusion, we note that the set Y_0 can have several connected components, as in the second picture of Fig. 2.1, whose closures do not intersect. Therefore, problems (2.2) can be considered on several sets, and the integrals in (2.3) are the sums of the integrals over these sets. In contrast, if Y_1 were represented as several connected sets with disjoint closures (as sets on a torus), then the homogenized problem would be a multiphase system of equations with convolutions. Such homogenized problems were received and justified in [22–24].

3. Laplace transform and proofs

To prove the Theorem 2.3, we will use the methods of Laplace transformation, which is isomorphism for relevant spaces [1]. Briefly, the exact definitions will be given here. This approach allows you to completely generalize the classic results for the thermal conductivity problem (problem (2.4) for $\mu = 0$) on problems with kernels $M(t)$ and $R(t)$, which are defined as elements of the space $L^1(0,\infty)$. This approach seems useful for other problems with convolutions, for example, for equations with fractional derivatives, parabolic transmission problems, and related problems with symmetric kernels, further details can be found in [8,16,27].

We fix real number ω and define the space $L^2_\omega(0,\infty;L^2(\Omega))$ as the set of functions from the space $L^2_{loc}(0,\infty;L^2(\Omega))$ for which the quantity

$$\|u\|_{L^2_\omega(0,\infty;L^2(\Omega))} = \|e^{-\omega t}u\|_{L^2(0,\infty;L^2(\Omega))}$$

is finite. The last equality defines a norm in the space $L^2_\omega(0,\infty;L^2(\Omega))$, with respect to which this space is complete in accordance with [1].

As in [1], let the space $E_\omega(L^2(\Omega))$ be the set of functions $W(z) = W(z_1 + iz_2)$ with values in $L^2(\Omega)$, which are continuous and holomorphic in the complex half-plane $\mathbb{C}_\omega = \{z \in \mathbb{C} : z = z_1 + iz_2,\ z_1 > \omega\}$, and for which the quantity

$$\|W\|^2_{E_\omega(L^2(\Omega))} = \int_{-\infty}^{\infty} \|W(\omega + iz_2)\|^2_{L^2(\Omega)}\, dz_2$$

is finite, where the integral is understood as the mean limit of the integral from K to $-K$ for $K \to \infty$. The last equality defines the norm for the space $E_\omega(L^2(\Omega))$.

The next version of the Paley-Wiener theorem was proved in [1].

Theorem 3.1. *For fixed* ω, *the Laplace transform*

$$\widehat{w(t)} = \int_0^\infty e^{-z\,t} w(t)\, dt = W(z)$$

is a bijective bicontinuous maps of $L^2_\omega(0,\infty;L^2(\Omega))$ *into* $E_\omega(L^2(\Omega))$.

In a similar way, we define the spaces $E_\omega(H_0^1(\Omega))$ and $E_\omega(H^{-1}(\Omega))$, for which an analogue of Theorem 3.1 also holds. In addition [1], the Laplace transform commutes with differentiations in space variables $x \in \Omega$ and maps convolution in $t \in (0,\infty)$ to pointwise multiplication with respect to $z \in \mathbb{C}$.

We will denote $V = \widehat{v}$, $Q = \widehat{q}$, $G = \widehat{g}$, and $F = \widehat{f}$ for Laplace transforms of solutions and data. Applying the Laplace transform to (2.4), we obtain

$$z\,(m - \mu\widehat{M}(z))V(z) - \mathrm{div}(\Lambda\nabla V(z)) = \mathbb{F}(z) \ \text{ in } \ \Omega, \quad V|_{\partial\Omega} = 0 \tag{3.1}$$

for $z \in \mathbb{C}$, where

$$\mathbb{F} = (r - \mu\widehat{R}(z))F(z) + (m - \mu\widehat{M}(z))w. \tag{3.2}$$

For fixed $z \in \mathbb{C}$, problem (3.1) is a boundary value problem for an elliptic equation with complex coefficients in the lower order terms. It is known [1], that the problem is solvable for all $z \in \mathbb{C}$ except, perhaps, a discrete subset in $\mathbb{C}$. Here, in order to prove the solvability of problem (2.4), it will be enough to separate from this discrete set using a priori estimates with constants independent of $z \in \mathbb{C}_0$.

The Laplace transform of problems (2.2) have the form

$$z\,\mu\, m_0 Q - \mathrm{div}_y(\lambda_0\nabla_y\, Q) = \mu\, m_0 \quad \text{in} \quad Y_0, \qquad Q|_{\overline{Y}_1} = 0, \tag{3.3}$$

$$z\,\mu\, m_0 G - \mathrm{div}_y(\lambda_0\nabla_y\, G) = \mu\, r_0 \quad \text{in} \quad Y_0, \qquad G|_{\overline{Y}_1} = 0. \tag{3.4}$$

Multiplying the first equation in (3.3) by $\overline{z}\,\overline{Q}$ and integrating over Y_0, we have

$$\int_{Y_0} |z\,Q|^2\,dy + \overline{z}\,\mu^{-1}m_0^{-1}\int_{Y_0} \left(\lambda_0\nabla_y\,Q, \nabla_y\,\overline{Q}\,\right)dy = \int_{Y_0} \overline{z}\,\overline{Q}\,dy. \tag{3.5}$$

Passing to the complex conjugate and using the symmetry of λ_0, we obtain

$$\int_{Y_0} |z\,Q|^2\,dy + z\,\mu^{-1}m_0^{-1}\int_{Y_0} \left(\lambda_0\nabla_y\,Q, \nabla_y\,\overline{Q}\,\right)dy = \int_{Y_0} z\,Q\,dy. \tag{3.6}$$

Summing up (3.5) and (3.6), for $z \in \mathbb{C}_0$ by ellipticity of λ_0 we get

$$\begin{aligned} \|zQ\|^2_{L^2(Y_0)} + z_1\alpha\,\mu^{-1}\beta^{-1}\|\nabla_y Q\|^2_{L^2(Y_0)} &\le \mathrm{Re}\int_{Y_0}(zQ)\,dy \ \le \int_{Y_0}|zQ|\,dy \\ &\le \|zQ\|_{L^2(Y_0)}|Y_0|^{1/2} \le (1/2)\|zQ\|^2_{L^2(Y_0)} + (1/2)|Y_0|. \end{aligned} \tag{3.7}$$

Thus, for $z \in \mathbb{C}_0$ we can conclude that

$$\|zQ\|^2_{L^2(Y_0)} \le |Y_0|. \tag{3.8}$$

Moreover, it follows from the definitions and equalities in (2.3) that

$$\widehat{M}\,|Y_1| \ = \int_{Y_0} m_0\,(zQ(z,y) - 1)\,dy \tag{3.9}$$

and

$$|\widehat{M}||Y_1| \le \int_{Y_0} m_0\,|zQ-1|\,dy \le \beta\,\|zQ\|_{L^2(Y_0)}|Y_0|^{1/2} + \beta\,|Y_0| \le 2\beta\,|Y_0|$$

according to (3.8). Therefore, the function $\widehat{M}$ is bounded on $\mathbb{C}_0$, since it is directly verified that problem (3.3) is uniquely solvable for each $z \in \mathbb{C}_0$ and such a solution is continuous and holomorphic by virtue of the Theorem 3.1 and the known properties of solutions to problems (2.2), established, for example, in [9].

Similarly, multiplying the first equation in (3.4) by $\overline{z}\,\overline{G}$, integrating over Y_0, and repeating the above proof, we obtain

$$|\widehat{R}\,||Y_1| \le \int_{Y_0} m_0\,|zG-(r_0/m_0)|\,dy \le (\beta+\alpha^{-1}\beta^2)\,|Y_0|.$$

Let us prove that the functions $\mathrm{Re}(-z\widehat{M})$ and $\mathrm{Re}(-\widehat{M})$ are positive on $\mathbb{C}_0$. Multiplying equation (3.3) by $z\overline{z}\,\overline{Q}$ and integrating over Y_0, we have

$$z\int_{Y_0}\overline{z}\,\overline{Q}(zQ-1)\,dy + \mu^{-1}m_0^{-1}\int_{Y_0}\big(\lambda_0\nabla_y zQ, \nabla_y \overline{z}\overline{Q}\,\big)dy = 0$$

and

$$z\int_{Y_0}(\overline{z}\,\overline{Q}-1)(zQ-1)\,dy + z\int_{Y_0}(zQ-1)\,dy$$
$$+\,\mu^{-1}m_0^{-1}\int_{Y_0}\big(\lambda_0\nabla_y zQ, \nabla_y \overline{z}\overline{Q}\,\big)dy = 0.$$

Therefore, using definition (3.9), we conclude that

$$-z\widehat{M}|Y_1| = z\big\|zQ-1\big\|^2_{L^2(Y_0)} + \mu^{-1}m_0^{-1}\int_{Y_0}\big(\lambda_0\nabla_y zQ, \nabla_y \overline{z}\overline{Q}\,\big)dy. \tag{3.10}$$

Similarly, by symmetry of λ_0 we can get the following conjugate equality

$$-\overline{z}\overline{\widehat{M}}|Y_1| = \overline{z}\big\|zQ-1\big\|^2_{L^2(Y_0)} + \mu^{-1}m_0^{-1}\int_{Y_0}\big(\lambda_0\nabla_y zQ, \nabla_y \overline{z}\overline{Q}\,\big)dy. \tag{3.11}$$

Thus, using the summation and ellipticity of λ_0, we deduce that

$$\mathrm{Re}(-z\widehat{M}) > 0, \qquad \mathrm{Re}(-\widehat{M}) > 0 \quad \text{for} \quad z \in \mathbb{C}_0 \tag{3.12}$$

by (3.10) and (3.11). Here, to check the last inequality, one should multiply (3.10) and (3.11) by $1/z = z_1/|z|^2 - iz_2/|z|^2$ and $1/\overline{z} = z_1/|z|^2 + iz_2/|z|^2$, respectively. In addition, it should be taken into account that the equality $zQ = 1$ cannot hold on Y_0 for $z \in \mathbb{C}_0$, since $zQ = 0$ on the boundary ∂Y_0 by the definition.

Multiplying equation (3.1) by $\overline{V}$ and integrating over Ω, we have

$$z(m-\mu\widehat{M})\int_{\Omega}|V|^2\,dx + \int_{\Omega}(\Lambda\nabla V, \nabla\overline{V})\,dx = \int_{\Omega}\mathbb{F}\overline{V}\,dx.$$

Passing to the complex conjugate and using the symmetry of Λ, we obtain

$$\overline{z}(m-\mu\overline{\widehat{M}})\int_\Omega |V|^2\,dx+\int_\Omega(\Lambda\nabla V,\nabla\overline{V})\,dx=\int_\Omega \overline{\mathbb{F}}V\,dx.$$

Summing up the equalities, for $z\in\mathbb{C}_0$ by ellipticity of Λ and (3.12) we get

$$z_1\alpha\,\|V\|^2_{L^2(\Omega)}+\alpha\,\|V\|^2_{H^1_0(\Omega)}\le \mathrm{Re}\int_\Omega(\mathbb{F}\overline{V})\,dx\le\|\mathbb{F}\,\|_{H^{-1}(\Omega)}\|V\|_{H^1_0(\Omega)}$$

and

$$\begin{aligned}\alpha\,\|V\|_{H^1_0(\Omega)}&\le\|\mathbb{F}\,\|_{H^{-1}(\Omega)}\\ &\le\big(\beta+(\beta+\alpha^{-1}\beta^2)\,|Y_0|\,|Y_0|^{-1}\big)\big(\|w\,\|_{H^{-1}(\Omega)}+\|F\,\|_{H^{-1}(\Omega)}\big)\end{aligned}$$

by definition (3.2), since m, r, $\widehat{M}$, and $\widehat{R}$ are bounded and $\mu\le 1$.

In fact, we obtain the following statement.

Lemma 3.1. *For every $z\in \mathrm{C}_0$, $f\in L^2(0,\infty;H^{-1}(\Omega))$ and $w\in H^{-1}(\Omega)$, the unique solution $V\in H^1_0(\Omega)$ to problem* (3.1) *exists and there is a positive constant C depending only on α, β, $|Y_0|$, and $|Y_1|$, such that*

$$\|V\|_{H^1_0(\Omega)}\le C\left(\|F\|_{H^{-1}(\Omega)}+\|w\|_{H^{-1}(\Omega)}\right). \tag{3.13}$$

The solution to problem (3.1) has some additional properties. Namely, as in [24] the continuity of the solution is valid according to the following assertion.

Lemma 3.2. *The solution $V=V(z)$ to problem* (3.1) *is continuous on $\mathbb{C}_0$.*

Proof. We fix $z_0\in\mathbb{C}_0$ and let $z\to z_0$. We introduce the notation

$$P(z)=z(m-\mu\widehat{M}(z)).$$

In the case, problems (3.1) at points z and z_0 can be written in the form

$$\begin{aligned}-\mathrm{div}(\Lambda\nabla V(z))+P(z)V(z)&=\mathbb{F}(z), && V(z)|_{\partial\Omega}=0,\\ -\mathrm{div}(\Lambda\nabla V(z_0))+P(z_0)V(\rho_0)&=\mathbb{F}(z_0), && V(z_0)|_{\partial\Omega}=0.\end{aligned}$$

We also denote $V_0(z)=V(z)-V(z_0)$. Subtracting the equations above and noting that $P(z_0)=P(z)+(P(z_0)-P(z))$, we obtain

$$-\mathrm{div}(\Lambda\nabla V_0(z))+P(z)V_0(z)=\mathbb{F}(z)-\mathbb{F}(z_0)-(P(z)-P(z_0))V(z_0). \tag{3.14}$$

Thus, repeating the proof of inequality (3.13), we have

$$\begin{aligned}\|V_0\|_{H^1_0(\Omega)}&\le C\,\|\mathbb{F}(z)-\mathbb{F}(z_0)\|_{H^{-1}(\Omega)}\\ &+|P(z)-P(z_0)|\,C\,\|V(z_0)\|_{H^{-1}(\Omega)}\to 0\end{aligned}$$

for $z\to z_0$, since $\mathbb{F}(z)$ and $P(z)$ are continuous by virtue of Theorem 3.1. □

Just as in [24], it is possible to prove that the solution $V(z)$ to problem (3.1) is holomorphic on $\mathbb{C}_0$. Formally, for this it suffices to divide (3.14) by $z - z_0$ and pass to the limit for $z \to z_0$, since $\mathbb{F}(z)$ and $P(z)$ are holomorphic. Thus, taking into account the integrability of $\mathbb{F}(z)$ from (3.2) in the sense of the definition of the space $E_0(H^{-1}(\Omega))$ for $w = 0$, we deduce the following statement.

Lemma 3.3. *Let $f \in L^2(0, \infty; H^{-1}(\Omega))$ and $w = 0$. Then, the unique solution $v \in L^2(0, \infty; H_0^1(\Omega))$ to problem (2.4) exists and there is a positive constant C depending only on α, β, $|Y_0|$, and $|Y_1|$, such that*

$$\|v\|_{L^2(0,\infty;H_0^1(\Omega))} \le C\|f\|_{L^2(0,\infty;H^{-1}(\Omega))}.$$

Using the linearity of problem (2.4), to complete the study of the solvability of the homogenized problems in convolutions, it remains to consider the case of arbitrary w and $f = 0$. In the case, the integrability condition $\mathbb{F}(z)$ from (3.2) in the sense of the definition of the space $E_0(H^{-1}(\Omega))$ is not satisfied in (3.13). But, one can also use the linearity of problem (2.4) and known estimates from [9].

Thus, consider the following auxiliary problem for the function $h = h(t, x)$:

$$\begin{aligned} &m\, h'_t \, - \, \mathrm{div}(\Lambda \nabla h) \, = \, 0 \quad \text{ in } \quad \Omega \times (0, \infty), \\ &h\big|_{t=0} = w \;\text{ in }\; \Omega, \quad h = 0 \;\text{ on }\; \partial\Omega \times (0, \infty). \end{aligned} \tag{3.15}$$

It is known that the unique solution $h \in L^2(0, \infty; H_0^1(\Omega))$ of this problem exists and $h'_t \in L^2(0, \infty; H^{-1}(\Omega))$. Indeed, multiplying the equation of problem (3.15) in $L^2(\Omega)$ by $\overline{h}$ and following [9], we obtain

$$\left(m \int_\Omega |h|^2 \, dx \right)'_t + 2 \int_\Omega (\Lambda \nabla h, \nabla \overline{h}\,) \, dx \; = \; 0.$$

Integrating this equality over time variable $\tau \in (0, t)$, we conclude that

$$\alpha \, \|h(t)\|^2_{L^2(\Omega)} + 2\alpha \int_0^t \|h\|^2_{H_0^1(\Omega)} \, d\tau \; \le \; \beta \, \|w\|^2_{L^2(\Omega)}.$$

Thus, passing to the limit with respect to t, we have $h \in L^2(0, \infty; H_0^1(\Omega))$.

Therefore, the solution to problem (3.15) satisfies the inclusion

$$m\, h'_t \; = \; \mathrm{div}(\Lambda \nabla h) \; \in L^2(0, \infty; H^{-1}(\Omega)).$$

Under the assumptions that $f = 0$, we introduce the notation $s = v - h$, where v and h are solutions to (2.4) and (3.15). Then s is a solution to the problem

$$\begin{aligned} &m\, s'_t \, - \, \mu M * (s'_t) \, - \, \mathrm{div}(\Lambda \nabla s) \, = \, \mu M * (h'_t) \quad \text{ in } \quad \Omega \times (0, \infty), \\ &s\big|_{t=0} = 0 \;\text{ in }\; \Omega, \quad s = 0 \;\text{ on }\; \partial\Omega \times (0, \infty). \end{aligned} \tag{3.16}$$

Thus, repeating the proof of Lemma 3.3, we have the following statement.

Lemma 3.4. *For $f = 0$ and $w \in L^2(\Omega)$, the solution $v \in L^2(0,\infty;H^1_0(\Omega))$ to problem* (2.4) *exists and is unique. Moreover, there is a positive constant C depending only on α, β, $|Y_0|$, and $|Y_1|$, such that*

$$\|v\|_{L^2(0,\infty;H^1_0(\Omega))} \leq C\,\|w\|_{L^2(\Omega)}.$$

Proof of Theorem 2.3. By virtue of Lemmas 3.3 and 3.4, it is necessary to prove that $v'_t \in L^2(0,\infty;H^{-1}(\Omega))$. For arbitrary f and w from Theorem 2.3, we rewrite problem (3.1) in the form: $V = 0$ on $\partial\Omega$ and

$$m\big(zV - w\big) = \big(1 - m^{-1}\mu\widehat{M}\,\big)^{-1}\Big(\big(r - \mu\widehat{R}\big)F + \operatorname{div}\big(\Lambda\nabla V\big)\Big) \quad \text{in} \quad \Omega, \qquad (3.17)$$

where the function $\big(1 - m^{-1}\mu\widehat{M}\,\big)^{-1}$ is defined, since from (3.8) and (3.12) we get

$$\begin{aligned} 1 \leq 1 - \operatorname{Re}\big(m^{-1}\mu\widehat{M}\,\big) &\leq \big|1 - \operatorname{Re}\big(m^{-1}\mu\widehat{M}\,\big)\big| \\ \leq \big|1 - m^{-1}\mu\widehat{M}\,\big| &\leq 1 + \big|m^{-1}\mu\widehat{M}\,\big| \leq 1 + 2\alpha^{-1}\big|Y_0\big|\big|Y_1\big|^{-1}. \end{aligned}$$

Consequently, reversing this inequality, we can conclude that

$$\big|\big(1 - m^{-1}\mu\widehat{M}\,\big)^{-1}\big| \leq 1.$$

Therefore, the function $\big(1 - m^{-1}\mu\widehat{M}\,\big)^{-1}$ is well defined and bounded on $\mathbb{C}_0$.

For $f \in L^2(0,\infty;H^{-1}(\Omega))$ and $w \in L^2(\Omega)$ the right-hand side of the equation in (3.17) belongs to $H^{-1}(\Omega)$ for $z \in \mathbb{C}_0$ and is integrable in the sense $E_0(H^{-1}(\Omega))$. In addition, the Laplace transform of the derivative $m\,v'_t$ for the solution to problem (2.4) coincides with the left-hand side of the equation in (3.17).

Thus, there is a constant C depending only on α, β, $|Y_0|$, and $|Y_1|$, such that

$$\|v'_t\|_{L^2(0,\infty;H^{-1}(\Omega))} \leq C\left(\|w\|_{L^2(\Omega)} + \|f\|_{L^2(0,\infty;H^{-1}(\Omega))}\right).$$

We fix a positive T. Then, the following inclusions $v \in L^2(0,T;H^1_0(\Omega))$ and $v'_t \in L^2(0,T;H^{-1}(\Omega))$ are valid. Thus, from the well-known embedding theorem given, for example, in [9], we conclude that $v \in C^0([0,T];L^2(\Omega))$.

Moreover, multiplying equation (3.1) by $\overline{z}\overline{V}$ and integrating over Ω, we get

$$(m - \mu\widehat{M})\int_\Omega |zV|^2\,dx + \overline{z}\int_\Omega (\Lambda\nabla V, \nabla\overline{V})\,dx = \int_\Omega \mathbb{F}\,\overline{z}\overline{V}\,dx.$$

Passing to the complex conjugate and summing up, for every $z \in \mathrm{C}_0$ we have

$$\alpha\,\|zV\|^2_{L^2(\Omega)} + z_1\alpha\,\|V\|^2_{H^1_0(\Omega)} \leq \operatorname{Re}\int_\Omega (\mathbb{F}\,\overline{z}\overline{V})\,dx \leq \|\mathbb{F}\,\|_{L^2(\Omega)}\|zV\|_{L^2(\Omega)}.$$

Thus, as in the proof of Lemma 3.1 we obtain the following statement.

Lemma 3.5. *For every $z \in \mathrm{C}_0$, $f \in L^2(0,\infty;L^2(\Omega))$ and $w \in L^2(\Omega)$, the unique solution $V \in H^1_0(\Omega)$ to problem* (3.1) *exists and there is a positive constant C depending only on α, β, $|Y_0|$, and $|Y_1|$, such that*

$$\|zV\|_{L^2(\Omega)} \leq C\left(\|F\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}\right).$$

For $w = 0$ the Laplace transform of v'_t for the solution to problem (2.4) is equal to zV. Thus, as in the proof of Lemma 3.3 we deduce the following assertion.

Lemma 3.6. *Let $f \in L^2(0,\infty; L^2(\Omega))$ and $w = 0$. Then, the unique solution $v \in L^2(0,\infty; H_0^1(\Omega))$ to problem (2.4) exists and $v'_t \in L^2(0,\infty; L^2(\Omega))$.*

For arbitrary $w \in H_0^1(\Omega)$ and $f = 0$, we again use the problem (3.15). Multiplying the equation of the problem by $\overline{h}'_t$ and following [9], we obtain

$$2\, m\int_\Omega |h'_t|^2\, dx + \left(\int_\Omega (\Lambda\nabla h, \nabla\overline{h}\,)\, dx\right)'_t = 0.$$

Integrating this equality over time variable $\tau \in (0,t)$, we conclude that

$$2\alpha\int_0^t \|h'_t\|^2_{L^2(\Omega)}\, d\tau \;\le\; \beta\,\|w\|^2_{H_0^1(\Omega)}.$$

Therefore, passing to the limit with respect to t, we have $h'_t \in L^2(0,\infty; L^2(\Omega))$.

Thus, returning to problem (2.4) and using Lemma 3.6, we conclude that there is a constant C depending only on α, β, $|Y_0|$, and $|Y_1|$, such that

$$\|v'_t\|_{L^2(0,\infty;L^2(\Omega))} \le C\left(\|w\|_{H_0^1(\Omega)} + \|f\|_{L^2(0,\infty;L^2(\Omega))}\right).$$

This inequality completes the proof of Theorem 2.3. □

Remark 3.1. The last inequality, representation (3.17) and elliptic regularity imply that $v \in L^2(0,\infty; H^2(\Omega))$. In fact, for example, for $f \in C_0^\infty((0,T)\times\Omega)$ and $w = 0$ one can repeat the proof of Lemma 3.5 and obtain that any time derivative of the solution is estimated by a suitable time derivative of f. Thus, using successively the representation (3.17) and elliptic regularity for solutions of problem (2.4), one can conclude that $v \in H^\sigma(0,\infty; H^{2\sigma}(\Omega))$ for arbitrarily large integers σ.

Proof of Lemma 2.1. As in the proof of (3.15), we multiply the first equation of problem (2.2) in $L^2(Y)$ by $\overline{q}$, then

$$2\alpha\int_0^t \|\nabla q\|^2_{L^2(Y)}\, d\tau \;\le\; \mu\beta\,\|1\|^2_{L^2(Y_0)}. \tag{3.18}$$

Therefore, passing to the limit with respect to t and using the Poincare inequality, we have $\|q\|^2_{L^2(0,\infty;L^2(Y))} \le C\mu$. Similarly, we can obtain the inequality for g. □

It is known [3, 5] that these inequalities imply the estimates

$$\|q_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} + \|g_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} \le C\mu \tag{3.19}$$

for $0 < \varepsilon \le \varepsilon_0$ with an appropriate ε_0, where the constant C may depend on Ω, $q_\varepsilon = q(t, x/\varepsilon)$ and $g_\varepsilon = g(t, x/\varepsilon)$ are defined in Theorem 2.1.

Proof of Theorem 2.2. Using the triangle inequality $\|a\|^2 \le 2\|a-b\|^2+2\|b\|^2$ and Theorem 2.1, we obtain

$$\begin{aligned} &\|u-v\|^2_{C^0([0,T];L^2(\Omega_1^\varepsilon))} + \mu\,\|u-v\|^2_{C^0([0,T];L^2(\Omega_0^\varepsilon))} \\ &\quad\le C\,\varepsilon + 2\,\mu\,\big\|q_\varepsilon * (v_t') - g_\varepsilon * f\big\|^2_{C^0([0,T];L^2(\Omega))}\,. \end{aligned}$$

To estimate the last term, we can use inequality (3.19) and Lemma 4.5 from [24]. In accordance with this lemma, for example, we get

$$\big\|(q_\varepsilon * (v_\tau'))(t)\big\|^2_{L^2(\Omega)} \le \|q_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))}\,\big\|v_t'\big\|^2_{L^2(0,T;L^\infty(\Omega))} \le C\mu, \tag{3.20}$$

since v is sufficiently regular according to [24] or Remark 3.1. Thus, we have

$$\|u-v\|^2_{C^0([0,T];L^2(\Omega_1^\varepsilon))} + \mu\,\|u-v\|^2_{C^0([0,T];L^2(\Omega_0^\varepsilon))} \le C\left(\varepsilon+\mu^2\right),$$

Under the conditions of the theorem, the solution v_0 is sufficiently regular also and $\varphi = v - v_0$ is the unique regular solution of the following problem

$$\begin{aligned} m\,\varphi_t' - \operatorname{div}(\Lambda\nabla\varphi) &= \mu\,M * (v_t') - \mu\,R * f \quad \text{in} \quad \Omega\times(0,\infty), \\ \varphi\big|_{t=0} &= 0 \;\;\text{in}\;\; \Omega, \quad \varphi = 0 \;\;\text{on}\;\; \partial\Omega\times(0,\infty). \end{aligned}$$

Therefore, repeating the proof of Theorem 2.3, we conclude that

$$\|v-v_0\|^2_{C^0([0,T];L^2(\Omega_1^\varepsilon))} + \|v-v_0\|^2_{C^0([0,T];L^2(\Omega_0^\varepsilon))} \le C\,\mu^2.$$

To complete the proof of Theorem 2.2, it remains to use the triangle inequality. □

Proof of Corollary 2.2. It is known [9] that the set $C_0^\infty((0,T)\times\Omega)$ is dense in the separable space $L^2(0,T;L^2(\Omega))$. Therefore, for every $f\in L^2(0,T;L^2(\Omega))$ there is a sequence $\{f_s\}_{s=1}^\infty \subset C_0^\infty((0,T)\times\Omega)$ such that

$$\|f-f_s\|^2_{L^2(0,T;L^2(\Omega))} \le \delta_s.$$

Here and below, δ_s is small enough for sufficiently large s.

For each s, we denote by u_s the solution of problem (2.1), in which f is replaced by f_s. Under the conditions of the corollary, let us check the following inequality

$$\|\varphi_s\|^2_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))} + \mu\,\|\varphi_s\|^2_{L^\infty(0,T;L^2(\Omega_0^\varepsilon))} \le C\delta_s,$$

where $\varphi_s = u - u_s$ and the constant C does not depend on ε, μ, and s. As in the proof of (3.15), we multiply by $\overline{\varphi}_s$ the equation of problem (2.1) for φ_s, then

$$\left(\int_\Omega m_\varepsilon^\mu\,|\varphi_s|^2 dx\right)_t' + 2\int_\Omega (\lambda^\varepsilon\nabla\varphi_s, \nabla\overline{\varphi}_s)\,dx = 2\int_\Omega r_\varepsilon^\mu\,\psi_s\,\overline{\varphi}_s dx,$$

where $\psi_s = f - f_s$. Integrating this equality over time variable $\tau \in (0,t)$, we get

$$\begin{aligned}\|\varphi_s(t)\|^2_{L^2(\Omega_1^\varepsilon)} + \mu\,\|\varphi_s(t)\|^2_{L^2(\Omega_0^\varepsilon)} \le\ & 2\,\alpha^{-1}\beta\,\|\psi_s\|_{L^1(0,T;L^2(\Omega_1^\varepsilon))}\|\varphi_s\|_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))}\\ &+\ 2\,\alpha^{-1}\beta\,\mu^{1/2}\|\psi_s\|_{L^1(0,T;L^2(\Omega_0^\varepsilon))}\,\mu^{1/2}\|\varphi_s\|_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))}\\ \le\ C\,\|\psi_s\|^2_{L^2(0,T;L^2(\Omega))} + (1/2)\|\varphi_s\|^2_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))} &+ (\mu/2)\|\varphi_s\|^2_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))}.\end{aligned}$$

Computing the essential supremum of the inequalities, we obtain exactly the verifiable inequality.

Similarly, we denote by v_0^s the solution of problem (2.4) for $\mu = 0$, in which f is replaced by f_s Then, as in the previous proof, one can conclude that

$$\|v_0 - v_0^s\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\delta_s.$$

For u_s, the estimate of Theorem 2.2 is satisfied, possibly with a constant depending on s, but independent of ε and μ. Thus we have, for example, that

$$\begin{aligned}\mu\,\|u - v_0\|^2_{L^\infty(0,T;L^2(\Omega_0^\varepsilon))} &= \mu\,\|u - u_s + u_s - v_0^s + v_0^s - v_0\|^2_{L^\infty(0,T;L^2(\Omega_0^\varepsilon))}\\ &\le C\delta_s + 2\,\mu\,\|u_s - v_0^s\|^2_{L^\infty(0,T;L^2(\Omega_0^\varepsilon))} \le C\delta_s + C_s\big(\,\varepsilon + \mu^2\,\big).\end{aligned}$$

For each positive δ one can choose and fix s so that $C\delta_s < \delta/2$. By virtue of the conditions of Corollary 2.2, one can find ε_δ and μ_δ such that $C_s(\,\varepsilon+\mu^2\,) < \delta/2$ for $\varepsilon < \varepsilon_\delta$ and $\mu < \mu_\delta$. Therefore, we get the inequality

$$\|u - v_0\|^2_{L^\infty(0,T;L^2(\Omega_1^\varepsilon))} + \mu\,\|u - v_0\|^2_{L^\infty(0,T;L^2(\Omega_0^\varepsilon))} < \delta$$

for all $\varepsilon < \varepsilon_\delta$ and $\mu < \mu_\delta$, which completes the proof by definition. □

Proof of Corollary 2.1. We denote by v_s the solution of problem (2.4), in which f is replaced by f_s Then, as in the proof of Theorem 2.3, we have

$$\|v - v_s\|^2_{L^\infty(0,T;L^2(\Omega))} + \big\|v' - v_s'\big\|^2_{L^2(0,T;L^2(\Omega))} \le C\delta_s$$

and the inequality from the previous proof holds.

Thus, we can repeat the previous proof, but with one remark. In the case under consideration, the inequality (3.20) cannot be applied, since the product of the functions $q_\varepsilon \in L^2(\Omega)$ and $v_t' \in L^2(\Omega)$ may not be defined as a function from $L^2(\Omega)$, and therefore the convolution may not be defined. Nevertheless, this product is defined as an element of the space $L^1(\Omega)$ and the inequality

$$\big\|(q_\varepsilon * (v_\tau'))(t)\big\|^2_{L^1(\Omega)} \le \|q_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))}\,\big\|v_t'\big\|^2_{L^2(0,T;L^2(\Omega))}$$

is valid. Thus, it is possible to write

$$\big\|(q_\varepsilon * (v - v_s)_\tau')(t)\big\|^2_{L^1(\Omega)} \le C\,\big\|v' - v_s'\big\|^2_{L^2(0,T;L^2(\Omega))} \le C\delta_s$$

and repeat the previous proof, which completes the proof of all statements. □

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