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HIGHER-ORDER OPTIMALITY CONDITIONS FOR DEGENERATE UNCONSTRAINED OPTIMIZATION PROBLEMS

Viktor Zadachyn*

Abstract. In this paper necessary and sufficient conditions of a minimum for the unconstrained degenerate optimization problem are presented. These conditions generalize the well-known optimality conditions. The new optimality conditions are presented in terms of polylinear forms and Hesse's pseudoinverse matrix. The results are illustrated by examples.The formulation and appearance of these conditions differ from high-order optimality conditions by other authors. The suggested representation of high-order optimality conditions makes them convenient for the evaluation of the convergence rate for unconstrained optimization methods in the case of a singular minimum point, for example, for the analysis of Newton's and quasi-Newton's methods.

Key words: unconditional optimization, degenerate minimum point, optimality conditions, multilinear form.

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1. Introduction

Unconstrained optimization is the aim of many papers, and it has a variety of applications (see, for example, [1–5]).Nevertheless, the existing numerical methods for solving the general unconstrained optimization problem up to the second order have a very low convergence rate in the case of degenerate problems [6–17] since for increasing the convergence rate, it is necessary to use derivatives of orders greater than two [6, 7]. At the same time, using derivatives of the 3^{rd} and 4^{th} orders makes a numerical method very time-consuming.

For the analysis of the convergence rate for unconstrained optimization methods in the case of a singular minimum point, it is necessary to have appropriate high-order optimality conditions.

A broad literature review on optimality conditions is presented in [18], therefore we will not go into details in this paper. In addition to the above review, many papers have been devoted to high-order optimality conditions (for example, [19, 20]). For the unconstrained optimization degenerate problem, high-order optimality conditions are formulated in [19], but the form of these conditions is not

*Viktor Zadachyn
Department of Information Systems, Simon Kuznets Kharkiv National University of Economics, Kharkiv, Ukraine Email: zadachinvm@gmail.com

convenient for application. In [20] high-order optimality conditions for unconstrained optimization have been also considered, but they are not convenient for usage as well.

This paper aims to represent generalized necessary and sufficient conditions of a minimum for unconstrained optimization degenerate problems, which improve to some extent mixed order necessary and sufficient conditions for a minimum proposed in [19] , but also differ from high-order optimality conditions suggested in [20]. The developed optimality conditions should be more convenient for evaluating the convergence rate of unconstrained optimization methods in the case of a singular minimum point.

2. Higher-order optimality conditions

Remind that the degenerate problem of unconstrained optimization is to find

$$\min f(x), \quad x \in R^n, \tag{2.1}$$

where $f(x)$ is of class C^p $(p \geq 4)$ under the assumption that a point $x^* \in R^n$ of local minimum of $f(x)$ is such that the Hessian matrix $f^{(2)}(x^*)$ is degenerate but is not zero identically.

Introduce the following notations:

$$R_1 = Ker(f^{(2)}(x^*)) = \{x \in R^n |\ f^{(2)}(x^*) \cdot x = 0\}; \quad R_2 = \{y \in R^n \mid y \perp x\}$$

is the orthogonal complement of R_1 (i.e., $R^n = R_1 \oplus R_2$); P is an orthogonal projector onto the subspace R_1; $P^{\perp}$ is an orthogonal projector onto the subspace R_2; $f^{(l)}(x^*)$ is an l-th derivative of $f(x)$ at $f^{(l)}(x^*) \cdot \left[u^i, v^{l-i}\right]$ is a multilinear form of l arguments u, $v \in R^n$ (the superscripts i and l-i indicate the multiplicity of occurrences of the corresponding argument). Notice that the value of symmetric multilinear form is invariant concerning various permutations of arguments.

Denote by $R^{(n)^{p/2}}$ the space of $\left(\frac{p}{2}\right)$-dimensional arrays of the dimension $n \times n \times \ldots \times n$. Then $f^{\left(\frac{p}{2}+1\right)}(x^*)$ can be considered as a linear mapping $f^{\left(\frac{p}{2}+1\right)}(x^*) : R^{(n)^{p/2}} \to R^n$; the mapping $(f^{\left(\frac{p}{2}+1\right)}(x^*))^T : R^n \to R^{(n)^{p/2}}$ is a conjugate to $f^{\left(\frac{p}{2}+1\right)}(x^*)$ linear mapping. The mapping $f^{(p)}(x^*)$ can be considered as a linear mapping $f^{(p)}(x^*) : R^{(n)^{p/2}} \to R^{(n)^{p/2}}$, i.e. the value of the multilinear form $f^{(p)} \cdot (x^*)\,[u^p] = U^T f^{(p)}(x^*)\,U$, where $U \in R^{(n)^{p/2}}$, is a $\left(\frac{p}{2}\right)$ — dimensional matrix with entries $U_{i,j,\ldots,k} = u_i u_j \cdots u_k$.

Remark also that if $(f^{(2)}(x^*))^+$ is a pseudoinverse matrix [21] to $f^{(2)}(x^*)$, then

$$P = I - \ (f^{(2)}(x^*))^+ f^{(2)}(x^*)\,, \quad P^{\perp} \ = (f^{(2)}(x^*))^+ f^{(2)}(x^*). \tag{2.2}$$

Theorem 2.1. *(generalized necessary condition for minimum). Let $f(x)$ be a function such that*

- *f attains at point $x^* \in R^n$ a local minimum;*

- *f is p times ($p \geq 4$, p is even) continuously differentiable in the neighborhood $V(x^*)$ of x^*;*
- *for all $u \in R^n$*
$$f^{(2l)}(x^*)\left[\ (Pu)^{2l}\right] = 0, \tag{2.3}$$
for $l = 1, \dots, \left(\frac{p}{2}-1\right)$.

Then for all $u \in R^n$

$$f^{(1)}(x^*) = 0, \quad f^{(2)}(x^*)\left[u^2\right] \geq 0; \tag{2.4}$$

$$f^{(2)}(x^*)\left[\ \left(P^{\perp}u\right)^2\right] \geq m_2||P^{\perp}u||^2; \tag{2.5}$$

$$\begin{gathered} f^{(2l+1)}(x^*)\left[\ (Pu)^{2l+1}\right] = 0, \\ for\ l = 1, \dots, \left(\tfrac{p}{2}-1\right),\ f^{(p)}(x^*)\left[(Pu)^p\right] \geq 0; \end{gathered} \tag{2.6}$$

$$f^{(l+1)}(x^*)\left[\ \left(P^{\perp}u\right),\ (Pu)^l\right] = 0, for\ l = 1, \dots, \left(\frac{p}{2}-1\right); \tag{2.7}$$

$$\left(f^{(p)}(x^*) - \frac{p!}{2\left(\left(\frac{p}{2}\right)!\right)^2}(f^{\left(\frac{p}{2}+1\right)}(x^*))^T(f^{(2)}(x^*))^+(f^{\left(\frac{p}{2}+1\right)}(x^*))\right)\left[(Pu)^p\right] \geq 0, \tag{2.8}$$

where $m_2 > 0$.

Proof. The conditions (2.4) are well known. The condition (2.5) means that the matrix $f^{(2)}(x^*)$ is not identically zero. Moreover, $m_2 > 0$ is equal to a minimal nonzero eigenvalue of the matrix $f^{(2)}(x^*)$. The conditions (2.6) and (2.7) follow from Theorem 2.1 [19] that was proved for the case of Hilbert space. Additionally, in Theorem 2.1 [19] it was proved that under condition (2.3) the following inequality

$$\begin{aligned} F_0(x^*, u) \equiv{}& \frac{1}{2}f^{(2)}(x^*)\left[\ \left(P^{\perp}u\right)^2\right] \\ &+ \frac{1}{\left(\frac{p}{2}\right)!}f^{\left(\frac{p}{2}+1\right)}(x^*)\left[\left(P^{\perp}u\right), (Pu)^{p/2}\right] + \frac{1}{p!}f^{(p)}(x^*)\left[(Pu)^p\right] \geq 0 \end{aligned} \tag{2.9}$$

holds for all $u \in R^n$.

Taking into account (2.2), we can rewrite (2.9) as follows

$$\begin{aligned} F_0(x^*, u) ={}& \frac{1}{2}f^{(2)}(x^*)\left[\left(P^{\perp}u + \frac{1}{\left(\frac{p}{2}\right)!}(f^{(2)}(x^*))^+(f^{\left(\frac{p}{2}+1\right)}(x^*))\left[(Pu)^{p/2}\right]\right)^2\right] \\ &+\frac{1}{p!}\left(f^{(p)}(x^*) - \frac{p!}{2\left(\left(\frac{p}{2}\right)!\right)^2}(f^{\left(\frac{p}{2}+1\right)}(x^*))^T(f^{(2)}(x^*))^+(f^{\left(\frac{p}{2}+1\right)}(x^*))\right)\left[(Pu)^p\right]. \end{aligned} \tag{2.10}$$

Since $F_0(x^*, u) \geq 0$ for all $u \in R^n$, consider decomposition $u = u_1 + u_2$, where $u_1 = Pu \in R_1$, $u_2 = P^{\perp}u = -\frac{1}{(\frac{p}{2})!}(f^{(2)}(x^*))^{+}(f^{\left(\frac{p}{2}+1\right)}(x^*))\left[(Pu)^{p/2}\right] \in R_2$. Then (2.8) follows from (2.4), (2.9), (2.10). □

Corollary 2.1. *(generalized necessary conditions for a minimum of* 4^{th} *order). Let* $f(x)$ *be a function such that it attains at point* $x^* \in R^n$ *a local minimum and is four times continuously differentiable in the neighborhood of* x^**.*

Then, for all $u \in R^n$

$$f^{(1)}(x^*) = 0,\ f^{(2)}(x^*)\left[u^2\right] = f^{(2)}(x^*)\left[\left(P^{\perp}u\right)^2\right] \geq 0; \tag{2.11}$$

$$f^{(2)}(x^*)\left[\left(P^{\perp}u\right)^2\right] \geq m_2||P\perp u||^2;$$

$$f^{(3)}(x^*)\left[(Pu)^3\right] = 0, f^{(4)}(x^*)\left[(Pu)^4\right] \geq 0; \tag{2.12}$$

$$\left(f^{(4)}(x^*) - 3\ (f^{(3)}(x^*))^T(f^{(2)}(x^*))^{+}(f^{(3)}(x^*))\right)[(Pu)^4] \geq 0, \tag{2.13}$$

where $m_2 > 0$.

Theorem 2.2. *(generalized sufficient minimum condition). Let* $f(x)$ *be a* p *times* $(p \geq 4,\ p$ *is even) continuously differentiable function in the neighborhood* $V(x^*)$ *of point* x^**at which the conditions (2.3)–(2.7) are satisfied and for all* $u \in R^n$

$$\left(f^{(p)}(x^*) - \frac{p!}{2\left((\frac{p}{2})!\right)^2}(f^{\left(\frac{p}{2}+1\right)}(x^*))^T(f^{(2)}(x^*))^{+}(f^{\left(\frac{p}{2}+1\right)}(x^*))\right) \times [(Pu)^p] \geq m_p||Pu||^p, \tag{2.14}$$

where $m_p > 0$.

Then x^* *is a point of strict local minimum of the function* $f(x)$ *and for all* x *from the sufficiently small neighborhood* $V(x^*)$ *the following inequality*

$$f(x) - f(x^*) \geq m_0 \cdot (||P\perp v||^2 + ||Pv||^p), \tag{2.15}$$

where $v = x - x^*$ *and* $m_0 > 0$, *holds.*

Proof. Since the function $f(x)$ is p times continuously differentiable in the neighborhood $V(x^*)$, according to the Taylor series expansion we have the following:

$$\begin{aligned} f(x) - f(x^*) &= \sum_{l=1}^{p} \frac{1}{l!} f^{(l)}(x^*)\left[(v)^l\right] + O(||v||^{p+1}) \\ &= f^{(1)}(x^*)[v] + \frac{1}{2} f^{(2)}(x^*)\left[\left(P^{\perp}v\right)^2\right] \\ &\quad + \sum_{l=3}^{p} \frac{1}{l!} \sum_{i=0}^{l} C_l^i f^{(l)}(x^*)\left[\left(P^{\perp}v\right)^{l-i}, (Pv)^i\right] + O(||v||^{p+1}), \end{aligned}$$

for all $x \in V(x^*)$, where $v = x - x^*$. Taking into account the conditions (2.3)–(2.7), in a sufficiently small neighborhood $V(x^*)$ the following equality

$$\begin{aligned} f(x) - f(x^*) &= \frac{1}{2} f^{(2)}(x^*)\left[\left(P^{\perp}v\right)^2\right] \\ &+ \frac{1}{\left(\frac{p}{2}\right)!} f^{\left(\frac{p}{2}+1\right)}(x^*)\left[\left(P^{\perp}v\right),\ (Pv)^{\frac{p}{2}}\right] \\ &+ \frac{1}{p!} f^{(p)}(x^*)\left[(Pv)^p\right] \\ &+ O(||P^{\perp}v||^3) + O(||P^{\perp}v|| \cdot ||Pv||^{\frac{p}{2}+1}) \\ &+ O(||P^{\perp}v||^2 \cdot ||Pv||^{\frac{p}{2}}) + O(||v||^{p+1}) \end{aligned}$$

holds. Because of (2.9) and (2.10), we have

$$\begin{aligned} f(x) - f(x^*) &= \frac{1}{2} f^{(2)}(x^*)\left[\left(P^{\perp}v + \frac{1}{\left(\frac{p}{2}\right)!}(f^{(2)}(x^*))^{+}(f^{\left(\frac{p}{2}+1\right)}(x^*))\left[(Pv)^{\frac{p}{2}}\right]\right)^2\right] \\ &+ \frac{1}{p!}\left(f^{(p)}(x^*) - \frac{p!}{2\left(\left(\frac{p}{2}\right)!\right)^2}(f^{\left(\frac{p}{2}+1\right)}(x^*))^{T}(f^{(2)}(x^*))^{+}(f^{\left(\frac{p}{2}+1\right)}(x^*))\right)\left[(Pv)^p\right] \\ &+ O(||P^{\perp}v||^3) + O(||P^{\perp}v|| \cdot ||Pv||^{\frac{p}{2}+1}) + O(||P^{\perp}v||^2 \cdot ||Pv||^{\frac{p}{2}}) + O(||v||^{p+1}). \end{aligned}$$

Thus, from (2.5) and (2.14) it follows that

$$\begin{aligned} f(x) - f(x^*) \geq \frac{m_2}{2}&\left\| P^{\perp}v + \frac{1}{\left(\frac{p}{2}\right)!}(f^{(2)}(x^*)) + (f^{\left(\frac{p}{2}+1\right)}(x^*))\left[(Pv)^{\frac{p}{2}}\right]\right\|^2 \\ &+ \frac{m_p}{p!}\|Pv\|^p - N_1\left\|P^{\perp}v\right\|^3 - N_2\left\|P^{\perp}v\right\| \cdot \|Pv\|^{\frac{p}{2}+1} \\ &- N_3\left\|P^{\perp}v\right\|^2 \cdot \|Pv\|^{\frac{p}{2}} - N_4\|v\|^{p+1}, \end{aligned} \tag{2.16}$$

where N_1, N_2, N_3, and N_4 are some positive constants.

Consider $x \in V(x^*)$ such that

$$\left\|P^{\perp}v\right\| \geq \frac{2}{\left(\frac{p}{2}\right)!}\left\|(f^{(2)}(x^*))^{+}\right\| \cdot \left\|f^{\left(\frac{p}{2}+1\right)}(x^*)\right\| \cdot \|Pv\|^{\frac{p}{2}}.$$

Then

$$\begin{aligned} &\left\| P^{\perp}v + \frac{1}{\left(\frac{p}{2}\right)!}(f^{(2)}(x^*))^{+}(f^{\left(\frac{p}{2}+1\right)}(x^*))\left[(Pv)^{\frac{p}{2}}\right]\right\| \geq \left\|P^{\perp}v\right\| \\ &- \left\|\frac{1}{\left(\frac{p}{2}\right)!}(f^{(2)}(x^*))^{+}(f^{\left(\frac{p}{2}+1\right)}(x^*))\left[(Pv)^{\frac{p}{2}}\right]\right\| \geq \frac{1}{2}\left\|P^{\perp}v\right\|, \end{aligned}$$

and hence (2.16) implies

$$\begin{aligned} f(x) - f(x^*) &\geq \frac{1}{2}\left(\frac{m_2}{8}\left\|P^{\perp}v\right\|^2 + \frac{m_p}{p!}\left\|Pv\right\|^p\right) \\ &\geq min(\frac{m_2}{16}, \frac{m_p}{2\cdot p!})\cdot\left(\left\|P^{\perp}v\right\|^2 + \left\|Pv\right\|^p\right) \end{aligned} \tag{2.17}$$

in a sufficiently small neighborhood $V(x^*)$.

Let $x \in V(x^*)$ is such that

$$\left\|P^{\perp}v\right\| < \frac{2}{\left(\frac{p}{2}\right)!}\left\|(f^{(2)}(x^*))^{+}\right\|\cdot\left\|f^{\left(\frac{p}{2}+1\right)}(x^*)\right\|\cdot\left\|Pv\right\|^{\frac{p}{2}}.$$

Then

$$\left\|Pv\right\|^{\frac{p}{2}} > \frac{\left(\frac{p}{2}\right)!}{2}\left\|(f^{(2)}(x^*))^{+}\right\|^{-1}\cdot\left\|f^{\left(\frac{p}{2}+1\right)}(x^*)\right\|^{-1}\cdot\left\|P^{\perp}v\right\|.$$

The inequality (2.5) implies that $\left\|(f^{(2)}(x^*))^{+}\right\| \leq \frac{1}{m_2}$ and, hence,

$$\begin{aligned} \left\|Pv\right\|^{\frac{p}{2}} &> \frac{\left(\frac{p}{2}\right)!}{2}\left\|(f^{(2)}(x^*))^{+}\right\|^{-1}\left\|f^{\left(\frac{p}{2}+1\right)}(x^*)\right\|^{-1}\left\|P^{\perp}v\right\| \\ &\geq \frac{\left(\frac{p}{2}\right)!}{2}m_2\left\|f^{\left(\frac{p}{2}+1\right)}(x^*)\right\|^{-1}\left\|P^{\perp}v\right\|. \end{aligned}$$

Then from (2.16) we obtain

$$\begin{aligned} f(x) - f(x^*) &\geq \frac{1}{2}\frac{m_p}{p!}\left\|Pv\right\|^p \\ &\geq \frac{1}{4}\frac{m_p}{p!}\left\|Pv\right\|^p + \frac{1}{4}\frac{m_p}{p!}\left(\frac{\left(\frac{p}{2}\right)!}{2}m_2\right)^2\left\|f^{\left(\frac{p}{2}+1\right)}(x^*)\right\|^{-2}\left\|P^{\perp}v\right\|^2 \\ &\geq min\left(\frac{m_p}{4\cdot p!}, \frac{1}{4}\frac{m_p}{p!}\left(\frac{\left(\frac{p}{2}\right)!}{2}m_2\right)^2\left\|f^{\left(\frac{p}{2}+1\right)}(x^*)\right\|^{-2}\right) \\ &\quad\times\left(\left\|P^{\perp}v\right\|^2 + \left\|Pv\right\|^p\right) \end{aligned} \tag{2.18}$$

in a sufficiently small neighborhood $V(x^*)$.

It is worth to emphasize that in our calculations given above (see, for instance, (2.18)), we implicitly assumed that $\left\|f^{\left(\frac{p}{2}+1\right)}(x^*)\right\| > 0$. In the case when

$$\left\|f^{\left(\frac{p}{2}+1\right)}(x^*)\right\| = 0$$

(i.e., $f^{\left(\frac{p}{2}+1\right)}(x^*) \equiv 0$), from (2.16) we conclude that

$$
\begin{aligned}
f(x) - f(x^*) &\geq \frac{1}{2}\left(\frac{m_2}{8}\left\|P^{\perp}v\right\|^2 + \frac{m_p}{p!}\left\|Pv\right\|^p\right) \\
&\geq min(\frac{m_2}{4}, \frac{m_p}{2\cdot p!})\cdot\left(\left\|P^{\perp}v\right\|^2 + \left\|Pv\right\|^p\right)
\end{aligned}
\tag{2.19}
$$

in a sufficiently small neighborhood $V(x^*)$,

Thus, according to (2.17) – (2.19) , for all $x \in V(x^*)$ different from x^*, there is a positive constant m_0 such that inequality (2.15) is fulfilled, i.e. x^* is a point of a strict local minimum of the function $f(x)$. □

Corollary 2.2. *(generalized sufficient condition for a minimum of the 4^{th} order). Let $f(x)$ be a four times continuously differentiable function in some neighborhood $V(x^*)$ of the point x^*, at which conditions (2.11) and (2.12) are satisfied, and for all $u \in R^n$*

$$
\left(f^{(4)}(x^*) - 3\,(f^{(3)}(x^*))^T(f^{(2)}(x^*))^{+}(f^{(3)}(x^*))\right)[(Pu)^4] \geq m_4\left\|Pu\right\|^4, \tag{2.20}
$$

where $m_4 > 0$.

Then x^* is a point of the strict local minimum of the function $f(x)$ and for all x from a sufficiently small neighborhood $V(x^*)$ the following inequality

$$
f(x) - f(x^*) \geq m_0\cdot\left(\left\|P^{\perp}v\right\|^2 + \left\|Pv\right\|^4\right), \tag{2.21}
$$

where $v = x - x^*$ and $m_0 > 0$, holds.

The given above generalized necessary and sufficient minimum conditions provide constructive optimality criteria for degenerate problem (2.1). We illustrate the naturalness of conditions (2.13) and (2.20) with the following examples.

Example 2.1. Consider the function $f(x) = \left(x_1 + x_2^2\right)^2$, $x \in R^2$. This function attains its minimum at the points of a set $X = \left\{x \in R^2 |\ x_1 = -x_2^2\right\}$.

Consider the point $x^* = (0,\ 0)^T \in X$. Then,

$$
f^{(2)}(x^*) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \left(f^{(2)}(x^*)\right)^{+} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
f^{(3)}(x^*) = (A \mid B), \quad f^{(4)}(x^*) = \begin{pmatrix} (C \mid C) \\ (C \mid D) \end{pmatrix},
$$

where $A = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, B = \begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 0 & 24 \end{pmatrix}$. The orthogonal projector $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and the orthogonal projector $P^{\perp} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The point x^* is not a point of strict local minimum, although the following condition

$$f^{(2)}(x^*)\left[\left(P^{\perp}u\right)^2\right]=2\left\|P^{\perp}v\right\|^2,\ f^{(4)}(x^*)\left[(Pu)^4\right]=24\cdot\|Pv\|^4,\ \forall\, u\in R^2$$

is satisfied. In addition,

$$\left(f^{(4)}(x^*)-3\,(f^{(3)}(x^*))^T(f^{(2)}(x^*))^{+}(f^{(3)}(x^*))\right)[(Pu)^4]=0,\ \forall\, u\in R^2.$$

Example 2.2. Consider the function $f(x)=x_1^2+x_1x_2^2+x_2^4,\ x\in R^2$. This function attains the minimal value at $x^*=(0,0)^T$, which is a point of the strict local minimum.

Then

$$f^{(2)}(x^*)=\begin{pmatrix}2&0\\0&0\end{pmatrix},\quad \left(f^{(2)}(x^*)\right)^{+}=\begin{pmatrix}\frac{1}{2}&0\\0&0\end{pmatrix},$$

$$f^{(3)}(x^*)=(A\mid B),\quad f^{(4)}(x^*)=\begin{pmatrix}(C\mid C)\\(C\mid D)\end{pmatrix},$$

where $A=\begin{pmatrix}0&0\\0&2\end{pmatrix},B=\begin{pmatrix}0&2\\2&0\end{pmatrix},C=\begin{pmatrix}0&0\\0&0\end{pmatrix},D=\begin{pmatrix}0&0\\0&24\end{pmatrix}$. The orthogonal projector $P=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ and the orthogonal projector $P^{\perp}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$.

For x^* the following conditions

$$f^{(2)}(x^*)\left[\left(P^{\perp}u\right)^2\right]=2\left\|P^{\perp}u\right\|^2,\ f^{(4)}(x^*)\left[(Pu)^4\right]=24\cdot\|Pu\|^4,\ \forall\, u\in R^2$$

$$\left(f^{(4)}(x^*)-3\,(f^{(3)}(x^*))^T(f^{(2)}(x^*))^{+}(f^{(3)}(x^*))\right)[(Pu)^4]]=18\cdot\|Pu\|^4,\ \forall\, u\in R^2$$

are satisfied.

Therefore, the condition (2.20) provides strictness of the minimum, while the condition $f^{(4)}(x^*)\left[(Pu)^4\right]\geq m\ \|Pu\|^4, \forall\, u\in R^2, m>0$, is not sufficient for this.

3. Conclusion

The suggested necessary and sufficient conditions of a minimum for unconstrained optimization degenerate problems generalize the known optimality conditions. The formulation and appearance of these conditions differ from the high-order optimality conditions proposed by other authors. Owing to the results obtained, the suggested optimality conditions can be used for the analysis of the convergence rate of unconstrained optimization methods in the case of a singular minimum point, for example, Newton's method and quasi-Newton's methods. These issues will be considered in future papers.

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