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FINDING THE ZEROS OF A HIGH-DEGREE POLYNOMIAL SEQUENCE

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Abstract. A 1-parameter initial-boundary value problem for a linear spatially 1-dimensional homogeneous degenerate wave equation, posed in a space-time rectangle, in case of strong degeneracy, was reduced to a linear integro-differential equation of convolution type (JODEA, 29(1) (2021), pp. 1–31). The former was then solved by applying the Laplace transformation, and the solution formula was inverted in the form of the Neumann series. The current study deals with an other approach to the inversion of the solution formula, based on invoking the Bromwich integral and the Cauchy residue theorem for the integrand. The denominator of the integrand being an infinite series with respect to rational functions of the complex variable, converges quite rapidly and can be approximated with finite series of m terms. Therefore finding the zeros of the approximated denominator reduces to finding the zeroes of a polynomial of degree $2m$. For the resulting polynomial sequence some numerical approaches have been applied.

Key words: degenerate wave equation, linear integro-differential equation of convolution type, Laplace transformation.

2010 Mathematics Subject Classification: 35L05, 35L35, 35L80.

1. Introduction and the problem formulation

The current study supplements our previous publication [1] dealing with the following 1-parameter simplified initial boundary value problem (IBVP) for the degenerate wave equation in the space-time rectangle $[0,T]\times[-1,+1]$

$$\begin{cases} \dfrac{\partial^2 u(t,x;\alpha)}{\partial t^2} = \dfrac{\partial}{\partial x}\left(a(x;\alpha)\,\dfrac{\partial u(t,x;\alpha)}{\partial x}\right), & (t,x)\in(0,T]\times(-1,+1)\,, \\ \left.\begin{aligned} u(t,-1;\alpha) &= h_2(t;\alpha) \\ u(t,+1;\alpha) &= h_1(t;\alpha) \end{aligned}\right\}, & t\in[0,T]\,, \\ \left.\begin{aligned} \dfrac{\partial u(0,x;\alpha)}{\partial t} &= \overset{**}{u}(x;\alpha) \\ u(0,x;\alpha) &= \overset{*}{u}(x;\alpha) \end{aligned}\right\}, & x\in[-1,+1]\,, \end{cases} \tag{1.1}$$

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where known control functions $h_1(t;\alpha), h_2(t;\alpha) \in \mathscr{C}^1[0,T] \bigcap \mathscr{C}^2(0,T)$ obey the compatibility conditions: $h_1(0;\alpha) = \overset{*}{u}(+1;\alpha)$, $h_1'(0;\alpha) = \overset{**}{u}(+1;\alpha)$, $h_2(0;\alpha) = \overset{*}{u}(-1;\alpha)$, and $h_2'(0;\alpha) = \overset{**}{u}(-1;\alpha)$, and the 1-parameter family of coefficient functions is defined as follows

$$a(x;\alpha) = |x|^\alpha, \qquad x \in [-1,+1], \tag{1.2}$$

the parameter $\alpha \in (0,2)$, and all the dependent and independent variables are nondimensional. One should refer to [1] to find out more details on the problem formulation.

In case of strong degeneracy, $\alpha \in (1,2)$, we split the original IBVP (1.1) posed in the space-time rectangle $[0,T] \times [-1,+1]$ into the derived IBVP$_2$ posed in the left space-time rectangle $[0,T] \times [-1,0])$ and the IBVP$_1$ posed in the right space-time rectangle $[0,T] \times [0,+1])$. Then we applied the method of separation of variables to find families of bounded solutions to the IBVP$_1$ and IBVP$_2$, having the continuous flux and depending on undetermined functions $h_3(t;\alpha), h_3(t;\alpha) \in \mathscr{C}^1[0,T] \bigcap \mathscr{C}^2(0,T]$. When implementing the continuity condition to the above families of solutions, we succeeded in deriving a linear integro-differential equation of convolution type with respect to the difference $h(t;\alpha) \equiv h_3(t;\alpha) - h_4(t;\alpha)$ of the required functions.

To solve the integro-differential equation of convolution type, we applied the Laplace transformation [3], producing for a function $f(t)$, $t \in [0,\infty)$, its transform as follows

$$F(\tau) = \mathfrak{L}\,[f(t)] := \int_0^\infty f(t)\,\mathrm{e}^{-\tau t}\,\mathrm{d}t\,, \qquad \tau = \xi + i\eta \in \mathbb{C}\,, \tag{1.3}$$

provided the original function $f(t)$ satisfies the known sufficient conditions for the image function $F(\tau)$ to exist, and obtained for the image $H(\tau;\alpha)$ of the original $h(t;\alpha)$ the following expression

$$\Delta H(\tau;\alpha) = \frac{R_2(\tau;\alpha) - R_1(\tau;\alpha)}{1 + Q_1(\tau;\alpha)}\,. \tag{1.4}$$

Generally, for finding the original function, the Bromwich integral

$$f(t) = \mathfrak{L}^{-1}\,[F(\tau)] = \frac{1}{2\pi i}\int_{\xi^*-i\infty}^{\xi^*+i\infty} F(\tau)\,\mathrm{e}^{+t\tau}\,\mathrm{d}\tau \tag{1.5}$$

is used, where $\Re\,\tau = \xi^*$ is a vertical straight line lying to the right of all the singularities of $F(\tau)$ (see Fig. 1.1, *a*). Nevertheless, in [1] we succeeded in inverting (1.4) in the form of the Neumann series. In the current study we will try to satisfy all the conditions for the application of the Bromwich integral to (1.4). First of all, this means finding the singularities of the right-hand side of (1.4). But, from our study [1], we know that the above singularities are nothing but the zeros of the denominator of the right-hand side of (1.4). So, from this there stems our concern in finding the zeros of the function $1 + Q_1(\tau;\alpha)$

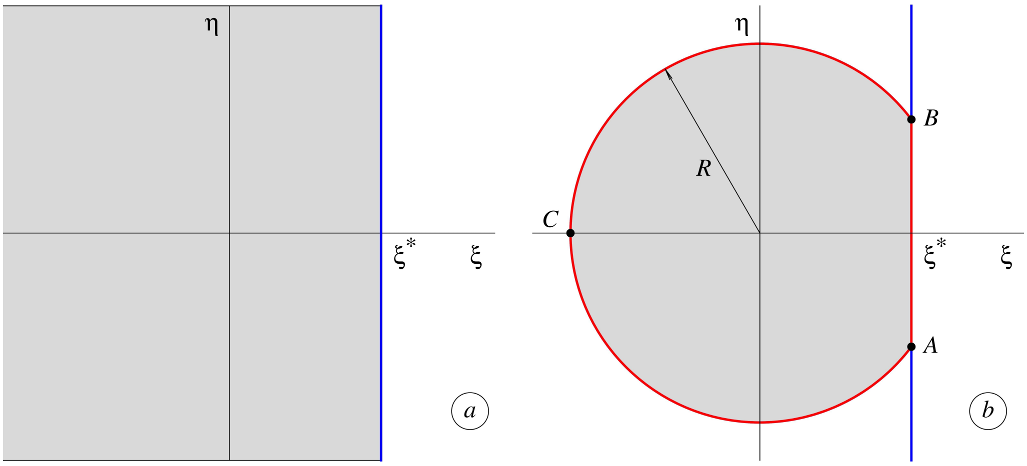

Fig. 1.1. All the singularities of the integrand of the Bromwich integral (1.5) lie in the half-plane (*gray color*) to the right of the path of integration $\tau = \xi^* + i\eta$, $\xi^* = \texttt{const}$, $\eta \in (-\infty, +\infty)$ (*dark blue*), or the Bromwich line (a); the oriented Bromwich contour $ABCA$ (*light red*), consists of the segment AB of the Bromwich line and the arc BCA of the circle of radius R centered at the origin; to apply the Cauchy residue theorem, the integrand must vanish at BCA when $R \to \infty$ and all the singularities of the integrand must lie inside $ABCA$ (b)

The current study is organized as follows.

In Sect. 2 we provide the most necessary knowledge of the function $Q_1(\tau;\alpha)$ in (1.4). In Sect. 3 we give the problem formulation. In Sect. 4 we outline in brief the method of solution. In Sect. 5 we discuss some most important results and observations.

2. Preliminaries

To clarify the complexity of the problem, we provide here only the most necessary knowledge of the function $Q_1(\tau;\alpha)$ in (1.4). It has the following series representation w.r.t. rational functions of $\tau = \xi + i\eta \in \mathbb{C}$

$$Q_1(\tau;\alpha) = C_\varrho \sum_{\mu=1}^{\infty} \sigma_{1,\mu}^{-\varrho} \frac{c_\mu \tau^2 + d_\mu}{\tau^2 + \sigma_{1,\mu}^2}\,, \tag{2.1}$$

where: 1) the coefficients in the numerators of the rational functions are

$$\begin{cases} c_\mu = +\dfrac{1}{\|Z_{1,\mu}\|^2}\, I_2 - \dfrac{1}{\|Z_{1,\mu}\|^2}\, I_0\,, \\[2ex] d_\mu = -\dfrac{\vartheta}{\|Z_{1,\mu}\|^2}\, I_1\,; \end{cases} \tag{2.2}$$

2) the functions $Z_{1,\mu}(x;\alpha)$, $|x| \in [0,1]$, and their norms are defined as follows

$$\begin{cases} Z_{1,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\, \mathtt{J}_{-\varrho}\Big(\mathring{s}\,|x|^{\frac{\theta}{2}}\Big) = \left(\dfrac{\mathring{s}}{2}\right)^{-\varrho} \displaystyle\sum_{\gamma=0}^{\infty} \dfrac{(-1)^{\gamma}\,|x|^{\gamma\theta}}{\gamma!\,\Gamma(1-\varrho+\gamma)} \left(\dfrac{\mathring{s}}{2}\right)^{2\gamma}, \\ \|Z_{1,\mu}\|^2 = \displaystyle\int_0^1 Z_{1,\mu}^2(x;\alpha)\,\mathrm{d}|x| = \dfrac{1}{\theta}\,\mathtt{J}^{\,2}_{-\varrho+1}(\mathring{s})\,; \end{cases} \tag{2.3}$$

3) $\mathtt{J}_{-\varrho}(s)$ is the Bessel function of the first kind and order $-\varrho$ [14]; 4) $\mathring{s} = s_{1,\mu}$, $\{s_{1,\mu}\}_{\mu=1}^{\infty}$ is the unbounded monotonically increasing sequence of the zeros of function $\mathtt{J}_{-\varrho}(s)$; 5) ν, θ, ϱ, $\sigma_{1,\mu}$, and C_{ϱ} are the following α-derived quantities

$$\nu = 1-\alpha\,, \quad \theta = 2-\alpha\,, \quad \varrho = \frac{\nu}{\theta} = \frac{1-\alpha}{2-\alpha}\,, \quad \sigma_{1,\mu} = \frac{\theta}{2}\,s_{1,\mu}\,, \quad C_{\varrho} = \frac{\theta^{\varrho}}{\Gamma(1-\varrho)}\,;$$

6) the definite integrals I_0, I_1, I_2 in (2.2) are calculated, applying the variable transformation

$$s = \mathring{s}\,|x|^{\frac{\theta}{2}} \equiv s_{1,\mu}\,|x|^{\frac{\theta}{2}}, \tag{2.4}$$

as follows

$$\begin{cases} I_0 \equiv \displaystyle\int_0^1 Z_{1,\mu}(x;\alpha)\,\mathrm{d}|x| = \dfrac{2}{\theta}\left(\dfrac{1}{\mathring{s}}\right)^{o+1} I_0^*\,, \\ I_1 \equiv \displaystyle\int_0^1 |x|^{\,\omega-\theta}\, Z_{1,\mu}(x;\alpha)\,\mathrm{d}|x| = \dfrac{2}{\theta}\left(\dfrac{1}{\mathring{s}}\right)^{\upsilon+1} I_1^*\,, \\ I_2 \equiv \displaystyle\int_0^1 |x|^{\,\omega} Z_{1,\mu}(x;\alpha)\,\mathrm{d}|x| = \dfrac{2}{\theta}\left(\dfrac{1}{\mathring{s}}\right)^{\upsilon+3} I_2^*\,, \end{cases} \tag{2.5}$$

where

$$\omega-\theta = 1+\epsilon\,, \quad \epsilon > 0\,, \qquad o = \frac{1}{\theta}\,, \quad \upsilon = \frac{2\epsilon+3}{\theta}\,, \qquad \vartheta \overset{(2.2)}{=} \omega\,[\omega-\theta+1]\,. \tag{2.6}$$

The transformed integrals in (2.5)

$$\begin{cases} I_0^* = \displaystyle\int_0^{\mathring{s}} s^{o}\,\mathtt{J}_{-\varrho}(s)\,\mathrm{d}s\,, \\ I_1^* = \displaystyle\int_0^{\mathring{s}} s^{\upsilon}\,\mathtt{J}_{-\varrho}(s)\,\mathrm{d}s\,, \\ I_2^* = \displaystyle\int_0^{\mathring{s}} s^{\upsilon+2}\,\mathtt{J}_{-\varrho}(s)\,\mathrm{d}s\,, \end{cases} \tag{2.7}$$

are calculated exactly, invoking the following recurrence formula [14] (see Sect. 7)

$$s^{-\varrho+1}\mathtt{J}_{-\varrho}(s) = \left[s^{-\varrho+1}\mathtt{J}_{-\varrho+1}(s)\right]'\,. \tag{2.8}$$

Thus, for the first integral (2.7) we have

$$I_0^* = \int_0^{\mathring{s}} s^o \, \mathrm{J}_{-\varrho}(s)\,\mathrm{d}s = \int_0^{\mathring{s}} s^{-\varrho+1}\, \mathrm{J}_{-\varrho}(s)\,\mathrm{d}s = \mathring{s}^{-\varrho+1}\, \mathrm{J}_{-\varrho+1}(\mathring{s})\,, \tag{2.9}$$

whereas for the second and thirds integrals (2.7), the recurrence formula [14] is applicable provided that proper values of the free parameter ϵ are taken as positive values produced by the formula [1]

$$\epsilon = -1 + k\,\theta, \qquad k \in \mathbb{N}\,. \tag{2.10}$$

3. Problem formulation

The series (2.1), representing the analytic function $Q_1(\tau;\alpha)$, converges quite rapidly in the whole plane $\tau = \xi + i\eta \in \mathbb{C}$, except for the simple poles $\tau_\mu^\mp = \mp\, i\sigma_{1,\mu}$, lying on the imaginary axis of the plane $\mathbb{C}$. Therefore we approximate the function $Q_1(\tau;\alpha)$ with the following finite series

$$Q_m(\tau;\alpha) = C_\varrho \sum_{\mu=1}^{m} \sigma_{1,\mu}^{-\varrho}\, \frac{c_\mu\,\tau^2 + d_\mu}{\tau^2 + \sigma_{1,\mu}^2}\,, \tag{3.1}$$

and pose the problem of finding the zeros of the function $1 + Q_m(\tau;\alpha)$ instead of finding the zeros of denominator of (1.4).

It is evident that the singular points $\tau_\mu^\mp$ do not contribute to the required set of the zeros of the function $1+Q_m(\tau;\alpha)$, hence w.l.o.g. we can neglect of these points, by multiplying the function $1 + Q_m(\tau;\alpha)$ by the product of the denominators of the rational functions in (3.1), to obtain the following polynomial of order $2m$

$$\begin{cases} P_{2m}(\tau;\alpha) \equiv \displaystyle\prod_{\gamma=1}^{m} \left(\tau^2 + \sigma_{1,\gamma}^2\right) \Big(1 + Q_m(\tau;\alpha)\Big) \\ \qquad = \displaystyle\prod_{\gamma=1}^{m} \left(\tau^2 + \sigma_{1,\gamma}^2\right) + C_\varrho \sum_{\mu=1}^{\infty} \left[\sigma_{1,\mu}^{-\varrho} \prod_{\substack{\gamma=1 \\ \gamma\neq\mu}}^{m} \left(\tau^2 + \sigma_{1,\gamma}^2\right) \Big(c_\mu\,\tau^2 + d_\mu\Big) \right], \end{cases} \tag{3.2}$$

being an analytic function in the whole plane $\tau \in \mathbb{C}$.

Based on the above transformation, we pose the following equivalent problem, to find the zeros of the polynomial $P_{2m}(\tau;\alpha)$ (3.2). According to the fundamental theorem of algebra, the above polynomial of order $2m$ has exactly $2m$ zeros, therefore, our concern is finding all $2m$ zeros of (3.2) and studying their dependence on m. To his end, we shall consider the sequences of the functions $1 + Q_m(\tau;\alpha)$ and the polynomials $P_{2m}(\tau;\alpha)$.

4. Method of solution

There are known a lot of techniques to find zeros of analytic functions, in particular, polynomials, in a region Ω, for example [5, 8, 9]. We choose a simple and robust method [8], and refer a reader to [8] for the details. Here, we only briefly outline the principal steps of the method:

1) covering the region Ω of interest with a triangular mesh;
2) evaluating the function for each nodal point of the mesh;
3) approxamating the function on each triangle by a bilinear function;
4) constructing piece-wise linear pairs of curves $C_{R,m} = \{\tau \in \Omega : U_m(t;\alpha) = 0\}$ and $C_{I,m} = \{\tau \in \Omega : V_m(t;\alpha) = 0\}$ for $Q_m(t;\alpha)$, $C_{R,2m} = \{\tau \in \Omega : U_{2m}(t;\alpha) = 0\}$ and $C_{I,2m} = \{\tau \in \Omega : V_{2m}(t;\alpha) = 0\}$ for $P_{2m}(t;\alpha)$, where

$$1 + Q_m(\tau;\alpha) = U_m(\tau;\alpha) + iV_m(\tau;\alpha)\,, \tag{4.1}$$

$$P_{2m}(\tau;\alpha) = U_{2m}(\tau;\alpha) + iV_{2m}(\tau;\alpha)\,; \tag{4.2}$$

5) storing the points where the curves $C_{R,m}$ and $C_{I,m}$, $C_{R,2m}$ and $C_{I,2m}$ cross;
6) refining the mesh if it is needed and repeating the above steps.

We choose a rectangular region Ω, cover it with a rectangular grid, and then divide each rectangular cell of the grid into two triangular subcells.

When evaluating Bessel functions of the first kind and non-integer orders and finding their zeros, we used algorithms [2, 12].

5. Numerical results

We present here numerical results for the following case

$$\alpha = \frac{5}{4} \quad \Rightarrow \quad \nu = -\frac{1}{4}\,, \quad \theta = \frac{3}{4}\,, \quad \varrho = -\frac{1}{3}\,, \quad \epsilon = \frac{1}{2}\,, \quad o = \frac{4}{3}\,, \quad \upsilon = \frac{16}{3}\,.$$

The first integral of (2.7), accordingly to (2.9), equals

$$I_0^* = \int_0^{\mathring{s}} s^o\, \mathrm{J}_{-\varrho}(s)\, \mathrm{d}s = \int_0^{\mathring{s}} s^{\frac{4}{3}}\, \mathrm{J}_{\frac{1}{3}}(s)\, \mathrm{d}s = \int_0^{\mathring{s}} \left[s^{\frac{4}{3}}\, \mathrm{J}_{\frac{4}{3}}(s)\right]' \mathrm{d}s = \mathring{s}^{\frac{4}{3}}\, \mathrm{J}_{\frac{4}{3}}(\mathring{s})\,, \tag{5.1}$$

whereas the second and third integrals of (2.7), due to (2.8), yield to

$$\left\{\begin{aligned} I_1^* &= \int_0^{\mathring{s}} s^\upsilon\, \mathrm{J}_{-\varrho}(s)\, \mathrm{d}s = \int_0^{\mathring{s}} s^{\frac{16}{3}}\, \mathrm{J}_{\frac{1}{3}}(s)\, \mathrm{d}s = \int_0^{\mathring{s}} s^4 \left[s^{\frac{4}{3}}\, \mathrm{J}_{\frac{4}{3}}(s)\right]' \mathrm{d}s \\ &= \mathring{s}^4\, \mathring{s}^{\frac{4}{3}}\, \mathrm{J}_{\frac{4}{3}}(\mathring{s}) - 4 \int_0^{\mathring{s}} s^2 \left[s^{\frac{7}{3}}\, \mathrm{J}_{\frac{7}{3}}(s)\right]' \mathrm{d}s \\ &= \mathring{s}^{\frac{16}{3}}\, \mathrm{J}_{\frac{4}{3}}(\mathring{s}) - 4\, \mathring{s}^2\, \mathring{s}^{\frac{7}{3}}\, \mathrm{J}_{\frac{7}{3}}(\mathring{s}) + 8 \int_0^{\mathring{s}} \left[s^{\frac{10}{3}}\, \mathrm{J}_{\frac{10}{3}}(s)\right]' \mathrm{d}s \\ &= \mathring{s}^{\frac{16}{3}}\, \mathrm{J}_{\frac{4}{3}}(\mathring{s}) - 4\, \mathring{s}^{\frac{13}{3}}\, \mathrm{J}_{\frac{7}{3}}(\mathring{s}) + 8\, \mathring{s}^{\frac{10}{3}}\, \mathrm{J}_{\frac{10}{3}}(\mathring{s})\,, \end{aligned}\right. \tag{5.2}$$

$$\begin{cases} I_2^* = \displaystyle\int_0^{\mathring{s}} s^{\upsilon+2}\,\mathtt{J}_{-\varrho}(s)\,\mathrm{d}s = \int_0^{\mathring{s}} s^{\frac{22}{3}}\,\mathtt{J}_{\frac{1}{3}}(s)\,\mathrm{d}s = \int_0^{\mathring{s}} s^6\left[s^{\frac{4}{3}}\,\mathtt{J}_{\frac{4}{3}}(s)\right]'\mathrm{d}s \\ \quad = \mathring{s}^6\,\mathring{s}^{\frac{4}{3}}\,\mathtt{J}_{\frac{4}{3}}(\mathring{s}) - 6\displaystyle\int_0^{\mathring{s}} s^4\left[s^{\frac{7}{3}}\,\mathtt{J}_{\frac{7}{3}}(s)\right]'\mathrm{d}s \\ \quad = \mathring{s}^{\frac{22}{3}}\,\mathtt{J}_{\frac{4}{3}}(\mathring{s}) - 6\,\mathring{s}^4\,\mathring{s}^{\frac{7}{3}}\,\mathtt{J}_{\frac{7}{3}}(\mathring{s}) + 24\displaystyle\int_0^{\mathring{s}} s^2\left[s^{\frac{10}{3}}\,\mathtt{J}_{\frac{10}{3}}(s)\right]'\mathrm{d}s \\ \quad = \mathring{s}^{\frac{22}{3}}\,\mathtt{J}_{\frac{4}{3}}(\mathring{s}) - 6\,\mathring{s}^{\frac{19}{3}}\,\mathtt{J}_{\frac{7}{3}}(\mathring{s}) + 24\,\mathring{s}^2\,\mathring{s}^{\frac{10}{3}}\,\mathtt{J}_{\frac{10}{3}}(\mathring{s}) - 48\displaystyle\int_0^{\mathring{s}}\left[s^{\frac{13}{3}}\,\mathtt{J}_{\frac{13}{3}}(s)\right]'\mathrm{d}s \\ \quad = \mathring{s}^{\frac{22}{3}}\,\mathtt{J}_{\frac{4}{3}}(\mathring{s}) - 6\,\mathring{s}^{\frac{19}{3}}\,\mathtt{J}_{\frac{7}{3}}(\mathring{s}) + 24\,\mathring{s}^{\frac{16}{3}}\,\mathtt{J}_{\frac{10}{3}}(\mathring{s}) - 48\,\mathring{s}^{\frac{13}{3}}\,\mathtt{J}_{\frac{13}{3}}(\mathring{s})\,. \end{cases} \tag{5.3}$$

Substituting the above integrals (5.1), (5.2), (5.3) into (2.5) yields to

$$\begin{cases} I_0 = \dfrac{8}{3}\dfrac{1}{\mathring{s}^4}\left[\mathring{s}^3\,\mathtt{J}_{\frac{4}{3}}(\mathring{s})\right], \\ I_1 = \dfrac{8}{3}\dfrac{1}{\mathring{s}^4}\left[\mathring{s}^3\,\mathtt{J}_{\frac{4}{3}}(\mathring{s}) - 4\,\mathring{s}^2\,\mathtt{J}_{\frac{7}{3}}(\mathring{s}) + \;\; 8\,\mathring{s}\,\mathtt{J}_{\frac{10}{3}}(\mathring{s})\right], \\ I_2 = \dfrac{8}{3}\dfrac{1}{\mathring{s}^4}\left[\mathring{s}^3\,\mathtt{J}_{\frac{4}{3}}(\mathring{s}) - 6\,\mathring{s}^2\,\mathtt{J}_{\frac{7}{3}}(\mathring{s}) + 24\,\mathring{s}\,\mathtt{J}_{\frac{10}{3}}(\mathring{s}) - 48\,\mathtt{J}_{\frac{13}{3}}(\mathring{s})\right], \end{cases}$$

and eventually to the coefficients (2.2)

$$\begin{cases} c_\mu = +\dfrac{2}{\mathring{s}^4\mathtt{J}^2_{\frac{4}{3}}(\mathring{s})}\left[\mathring{s}^3\,\mathtt{J}_{\frac{4}{3}}(\mathring{s}) - 6\,\mathring{s}^2\,\mathtt{J}_{\frac{7}{3}}(\mathring{s}) + 24\,\mathring{s}\,\mathtt{J}_{\frac{10}{3}}(\mathring{s}) - 48\,\mathtt{J}_{\frac{13}{3}}(\mathring{s})\right] \\ \qquad - \dfrac{2}{\mathring{s}^4\mathtt{J}^2_{\frac{4}{3}}(\mathring{s})}\left[\mathring{s}^3\,\mathtt{J}_{\frac{4}{3}}(\mathring{s})\right], \\ d_\mu = -\dfrac{15}{\mathring{s}^4\mathtt{J}^2_{\frac{4}{3}}(\mathring{s})}\left[\mathring{s}^3\,\mathtt{J}_{\frac{4}{3}}(\mathring{s}) - 4\,\mathring{s}^2\,\mathtt{J}_{\frac{7}{3}}(\mathring{s}) + \;\; 8\,\mathring{s}\,\mathtt{J}_{\frac{10}{3}}(\mathring{s})\right]. \end{cases}$$

First, we applied the method of finding the zeros, outlined briefly in Sect. 4, to the sequence of the functions $1+Q_m(t;\alpha)$ (3.1) and presented numerical results in Fig. 7.2 and 7.3. Then, we studied the sequence of the polynomials $P_{2m}(t;\alpha)$ (3.2) and presented numerical results in Fig. 7.4 and 7.5.

Contrary to our expectations, we could not succeed in comparing the obtained results for both sequences, since the number and position of zeros in Figs. 7.2, 7.3 and Figs. 7.4, 7.5 differ significantly. Careful examination of the surface plots for $U_m(t;\alpha)$, $V_m(t;\alpha)$, $|1+Q_m(t;\alpha)|$, $U_{2m}(t;\alpha)$, $V_{2m}(t;\alpha)$, and $|P_{2m}(t;\alpha)|$ revealed that the difference surely originated from the Gibbs phenomenon for the expansions of blending functions, used in [1], in series involving Bessel functions of the first kind. In the current study the Gibbs phenomenon manifested itself through the coefficients (2.2). Nevertheless, the Gibbs phenomenon did not manifest itself in the problem of finding the zeros of the polynomials $P_{2m}(t;\alpha)$. We think the reason for this is that the polynomials grow rapidly with distance from the point $\tau=0$, whereas the Gibbs phenomenon is negligibly small compared to this growth to produce false zeros and avoid the convergence of the zeros. Indeed, from Fig. 7.4 one could suppose, that when the order of the polynomials increases, the previously found zeros converge, that is their positions stop changing. Fig. 7.5 proved that the above assumption is correct.

Studying Figs. 7.2, we could not distinguish between correct and false zeros. Fortunately, for the case being discussed, the polynomial sequence gave the correct positions of the zeros for quite enough very-high orders, therefore we could succeed in separating the false zeros from the correct ones in Figs. 7.2, and 7.3. But generally, increasing the order of the polynomials eventually leads rapidly to overflow errors. Therefore, finding the zeros of the sequence of the functions $1 + Q_m(t;\alpha)$ needs proper curing. We believe that the method of Sect. 4 supplemented by some approaches, for example [4, 6, 7, 10], to the suppression of the Gibbs phenomenon, will work correctly.

6. Conclusions

In the current study we tried to estimate the possibility of using the Bromwich integral (1.5) for inverting (1.4), considering some approaches and some particular cases of the IBVP (1.1).

1. Approxamating the function $Q_1(\tau;\alpha)$ (2.1) by finite series $Q_m(\tau;\alpha)$ (3.1) simplifies the problem of finding zeros of the denominator $1+Q_1(\tau;\alpha)$ of (1.4), but, unfortunately, finite series $Q_m(\tau;\alpha)$ have happened to be very sensitive to the Gibbs phenomenon to produce false zeros.

2. Reducing the problem of finding the zeros of the functions $1+Q_m(\tau;\alpha)$ to the problem of finding the zeros of the polynomials $P_{2m}(\tau;\alpha)$ (3.2) avoids producing the false zeros, demonstrate convergence of the zeros, but generally could lead to overflow errors.

7. Corrections

1. In the expressions following (3.5) on p. 18 [1], $\sigma^{-\varrho}_{\varrho,\mu}$ should be replaced with $\sigma^{-\varrho}_{1,\mu}$.

2. In the expression for I_2^* on p. 28 [1], $\mathtt{J}_{\frac{5}{3}}(\mathring{s})$ should be replaced with $\mathtt{Z}_{\frac{5}{3}}(\mathring{s})$.

3. A generic notation $\mathtt{Z}_\varrho(s)$ introduced in Sect. 2.1 [1], revealed to be fragile in Appendix [1], since it implied that ϱ not only as the index of $\mathtt{Z}_\varrho(s)$ should be treated as $\mp\varrho$ for the Bessel functions $\mathtt{J}_{\mp\varrho}(s)$ of the first kind and orders $\mp\varrho$, respectively, but also as the exponents of the powers of $\mathring{s}$. Sometimes this implication could lead to confusion. In Sects. 2, 3 and 5 we replaced this generic notation with explicit denoting the order of the Bessel functions and the exponents of the powers of $\mathring{s}$.

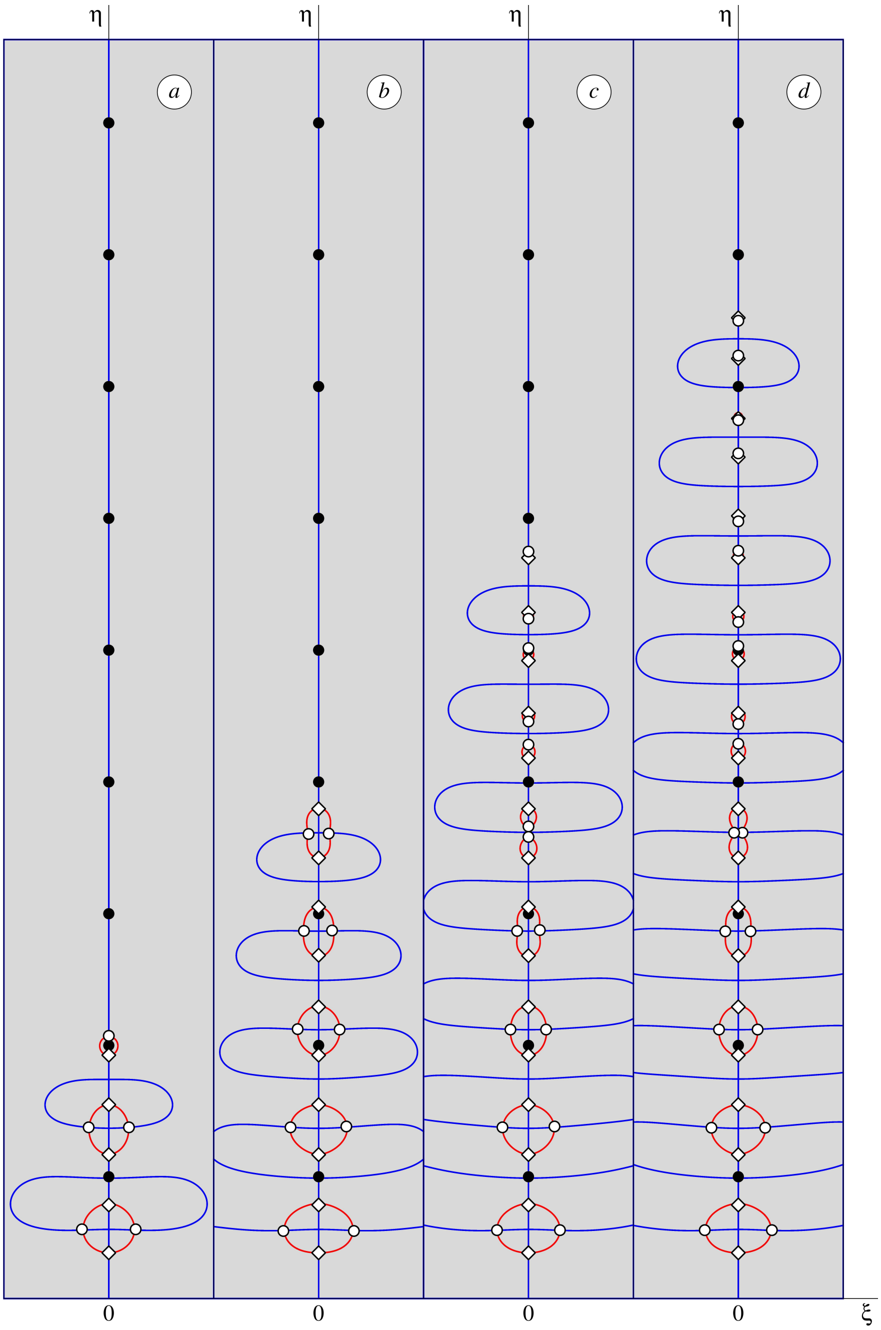

Fig. 7.2. The rectangular region $[-2.5, +2.5] \times [0.0, 25.0] \in \mathbb{C}$ in which the lines of zero values of the real $U_m(\tau;\alpha)$ (red color) and the imaginary $V_m(\tau;\alpha)$ (blue color) parts (4.1) of the functions $1 + Q_m(\tau;\alpha)$ (3.1) are drawn: $m = 5$ (a), $m = 10$ (b), $m = 15$ (c), $m = 20$ (d). Singular points of the function $Q_1(\tau;\alpha)$ (2.1) and the correct and false zeros of functions $Q_m(\tau;\alpha)$ are labeled as black and white discs, and white diamonds respectively

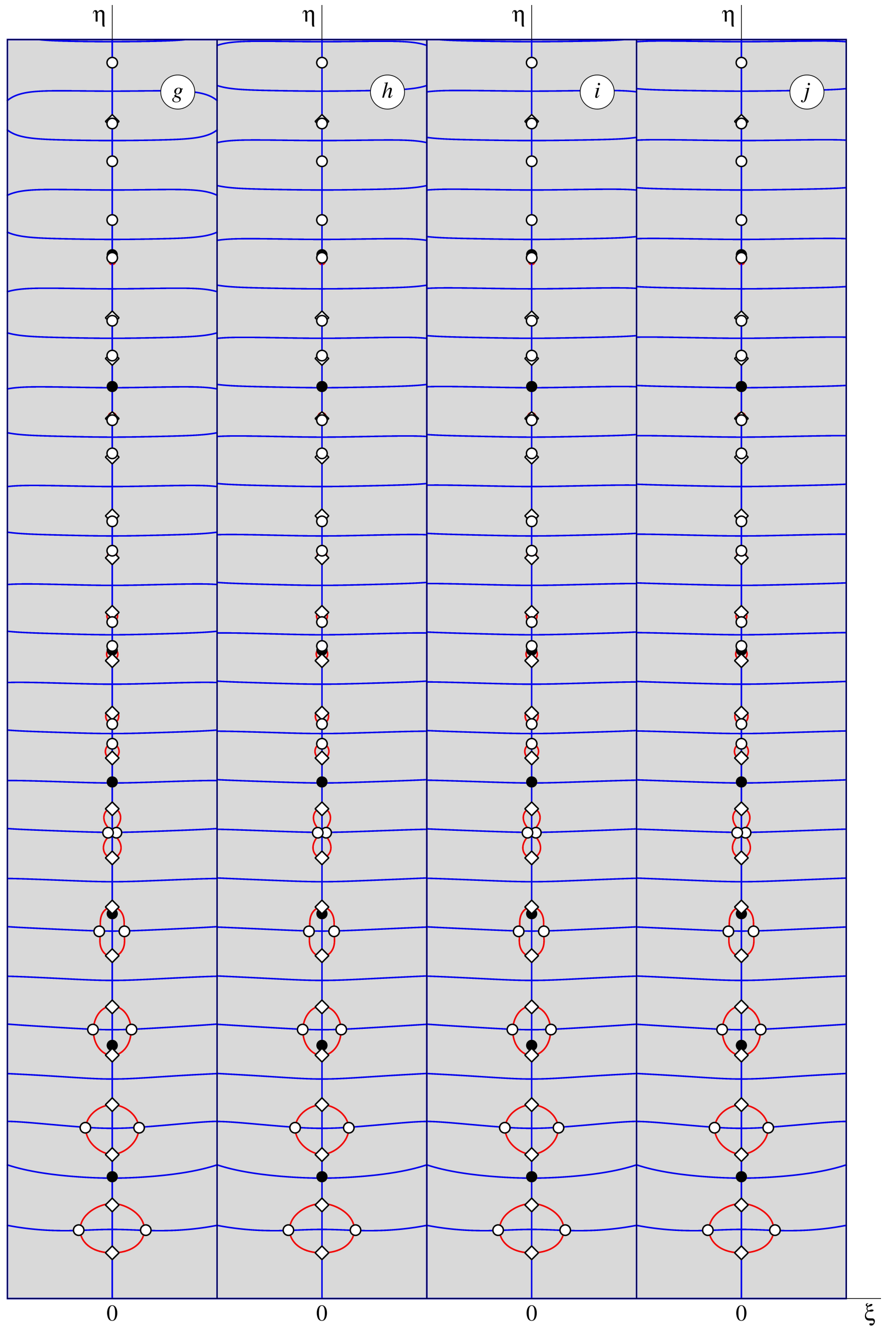

Fig. 7.3. The rectangular region $[-2.5, +2.5] \times [0.0, 25.0] \in \mathbb{C}$ in which the lines of zero values of the real $U_m(\tau;\alpha)$ (red color) and the imaginary $V_m(\tau;\alpha)$ (blue color) parts (4.1) of the functions $1 + Q_m(\tau;\alpha)$ (3.1) are drawn: $m = 35$ (g), $m = 40$ (h), $m = 45$ (i), $m = 50$ (j). Singular points of the function $Q_1(\tau;\alpha)$ (2.1) and the correct and false zeros of functions $Q_m(\tau;\alpha)$ are labeled as black and white discs, and white diamonds respectively

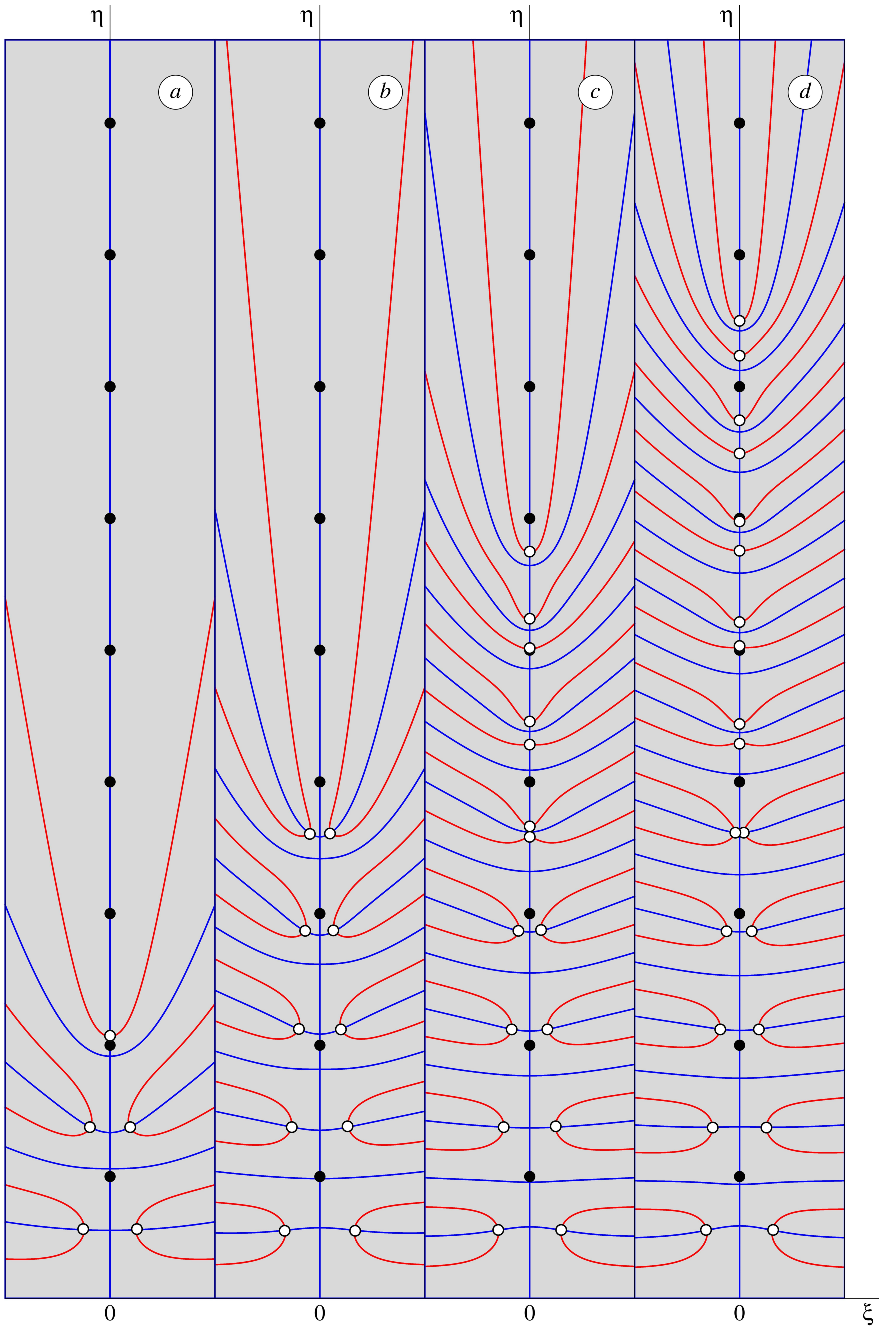

Fig. 7.4. The rectangular region $[-2.5, +2.5] \times [0.0, 25.0] \in \mathbb{C}$ in which the lines of zero values of the real $U_{2m}(\tau; \alpha)$ (red color) and the imaginary $V_{2m}(\tau; \alpha)$ (blue color) parts (4.2) of the polynomials $P_{2m}(\tau; \alpha)$ (3.2) are drawn: $m = 5$ (a), $m = 10$ (b), $m = 15$ (c), $m = 20$ (d). Singular points of the function $Q_1(\tau; \alpha)$ (2.1) and the zeros of polynomials $P_{2m}(\tau; \alpha)$ are labeled as black and white discs, respectively

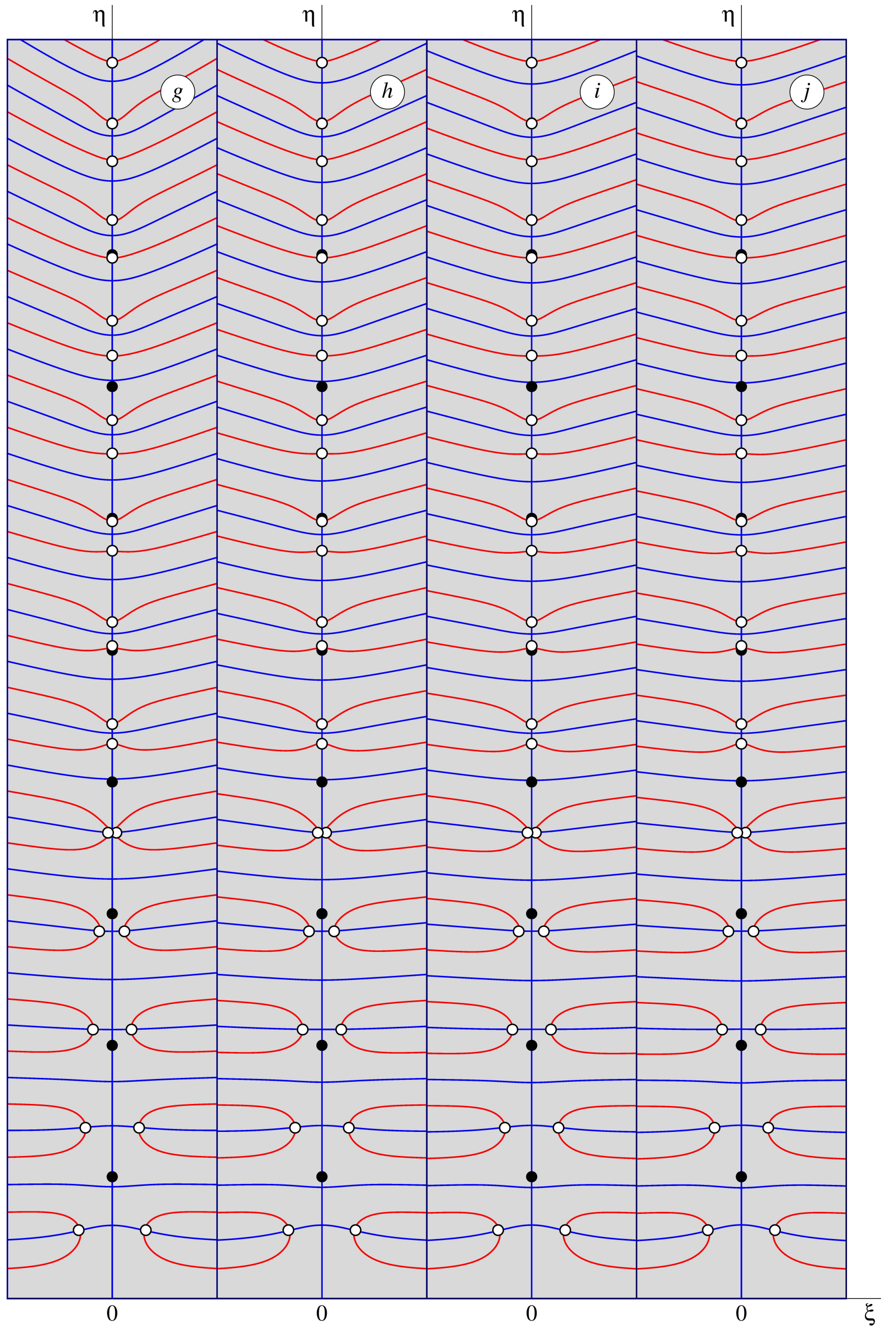

Fig. 7.5. The rectangular region $[-2.5, +2.5] \times [0.0, 25.0] \in \mathbb{C}$ in which the lines of zero values of the real $U_{2m}(\tau;\alpha)$ (red color) and the imaginary $V_{2m}(\tau;\alpha)$ (blue color) parts (4.2) of the polynomials $P_{2m}(\tau;\alpha)$ (3.2) are drawn: $m=35$ (g), $m=40$ (h), $m=45$ (i), $m=50$ (j). Singular points of the function $Q_1(\tau;\alpha)$ (2.1) and the zeros of polynomials $P_{2m}(\tau;\alpha)$ are labeled as black and white discs, respectively

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