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SHAPE STABILITY OF OPTIMAL CONTROL PROBLEMS IN COEFFICIENTS FOR COUPLED SYSTEM OF HAMMERSTEIN TYPE

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In this paper we consider an optimal control problem (OCP) for the coupled system of a nonlinear monotone Dirichlet problem with matrix-valued $L^\infty(\Omega;\mathbb{R}^{N\times N})$-controls in coefficients and a nonlinear equation of Hammerstein type, where solution nonlinearly depends on L^∞-control. Since problems of this type have no solutions in general, we make a special assumption on the coefficients of the state equations and introduce the class of so-called solenoidal admissible controls. Using the direct method in calculus of variations, we prove the existence of an optimal control. We also study the stability of the optimal control problem with respect to the domain perturbation. In particular, we derive the sufficient conditions of the Mosco-stability for the given class of OCPs.

Key words: nonlinear monotone Dirichlet problem, equation of Hammerstein type, control in coefficients, domain perturbation.

1. Introduction

The aim of this paper is to prove the existence result for an optimal control problem (OCP) governed by the system of a nonlinear monotone elliptic equation with homogeneous Dirichlet boundary conditions and a nonlinear equation of Hammerstein type, and to provide sensitivity analysis of the considered optimization problem with respect to the domain perturbations. As controls we consider the matrix of coefficients in the main part of the elliptic equation and a coefficient in the non-linear part of the Hammerstein equation. We assume that admissible controls are measurable and uniformly bounded functions from $L^\infty(\Omega;\mathbb{R}^{N\times N})\times L^\infty(\Omega)$.

Systems with distributed parameters and optimal control problems for systems described by PDE, nonlinear integral and ordinary differential equations have been widely studied by many authors (see for example [17, 21, 22]). However, systems which contain equations of different types and optimization problems associated with them are still less well understood. In general case including as well control and state constrains, such problems are rather complex and have no simple constructive solutions. The system, considered in the present paper, contains two equations: a nonlinear monotone elliptic equation with homogeneous Dirichlet boundary conditions and a nonlinear equation of Hammerstein type, which nonlinearly depends on the solution of the first object. The optimal control problem we study here is to minimize the discrepancy between a given distribution $z_d \in L^p(\Omega)$ and a solution of Hammerstein equation $z = z(\mathcal{U}, v, y)$, choosing appropriate coefficients $(\mathcal{U}, v) \in U_{ad} \times V_{ad}$, i.e.

$$I_\Omega(\mathcal{U}, v, y, z) = \int_\Omega |z(x) - z_d(x)|^p \, dx \longrightarrow \inf \tag{1.1}$$

subject to constrains

$$z + BF(v, y, z) = g \quad \text{in } \Omega, \tag{1.2}$$

$$-\operatorname{div}\big(\mathcal{U}(x)[(\nabla y)^{p-2}]\nabla y\big) + |y|^{p-2}y = f \quad \text{in } \Omega, \tag{1.3}$$

$$(\mathcal{U}, v) \in U_{ad} \times V_{ad}, \quad y \in W_0^{1,p}(\Omega), \tag{1.4}$$

where $U_{ad} \times V_{ad} \subset L^\infty(\Omega; \mathbb{R}^{N\times N}) \times L^\infty(\Omega)$ is a set of admissible controls, $B : L^q(\Omega) \to L^p(\Omega)$ is a positive linear operator and $F : L^\infty(\Omega) \times W_0^{1,p}(\Omega) \times L^p(\Omega) \to L^q(\Omega)$ is an essentially nonlinear and non-monotone operator.

Since the range of optimal control problems in coefficients is very wide, including as well optimal shape design problems, optimization of certain evolution systems, some problems originating in mechanics and others, this topic has been widely studied by many authors. Typically (see for instance [22, 24]), the most of optimal control problems in coefficients for linear elliptic equations have no solution in general. It turns out that this circumstance is the characteristic feature for the majority of optimal control problems in coefficients. To overcome this difficulty, in present article, by analogy with [10, 18, 20], we put some additional constrains on the set of admissible controls. Namely, we consider the matrix-valued controls from the so-called generalized solenoidal set. The elements of this set do not belong to any Sobolev space, but still are a little bit “more regular” then those from L^∞-class. We give the precise definition of such controls in Section 3 and prove that in this case the original optimal control problem admits at least one solution. It should be noticed that we do not involve the homogenization method and the relaxation procedure in this process.

In practice, the equations of Hammerstein type appear as integral or integro-differential equations. The class of integral equations is very important for theory and applications, since there are less restrictions put on smoothness of the desired solutions involved in comparison to those for the solutions of differential equations. It should be also mentioned here that solution uniqueness is not typical for equations of Hammerstein type or optimization problems associated with such objects (see [1]). Indeed, such property would require rather strong assumptions on operators B and F, which is rather restrictive in view of numerous applications (see [25]).

As was pointed above, the principal feature of this problem is the fact that an optimal solution for (1.1)–(1.4) does not exist in general. So here we have a typical situation for the general optimal control theory. Namely, the original control object is described by well-posed boundary value problem, but the associated optimal control problem is ill-posed and requires relaxation.

Since there is no good topology a priori given on the set of all open subsets of $\mathbb{R}^N$, we study the stability properties of the original control problem imposing some constraints on domain perturbations. Namely, we consider two types of domain perturbations: so-called topologically admissible perturbations (see Dancer [8]), and perturbations in the Hausdorff complementary topology (see Bucur and Zolesio [5]). The asymptotical behavior of sets of admissible quadruples Ξ_ε — controls and the corresponding states — under domain perturbation is described in detail in Section 4. In particular, we show that in this case the sequences of admissible solutions to the perturbed problems are compact with respect to the weak convergence in $L^\infty(D;\mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(D) \times L^p(D)$. Section 5 is devoted to the stability properties of optimal control problem (1.1)–(1.4) under the domain perturbation. Our treatment of this question is based on a new stability concept for optimal control problems (see for comparison [10, 11]). We show that Mosco-stable optimal control problems possess "good" variational properties, which allow using optimal solutions to the perturbed problems in "simpler" domains as a basis for the construction of suboptimal controls for the original control problem. As a practical motivation of this approach we want to point out that the "real" domain Ω is never perfectly smooth but contains microscopic asperities of size significantly smaller than characteristic length scale of the domain. So a direct numerical computation of the solutions of optimal control problems in such domains is extremely difficult. Usually it needs a very fine discretization mesh, which means an enormous computation time, and such a computation is often irrelevant. In view of the variational properties of Mosco-stable problems we can replace the "rough" domain Ω by a family of more"regular" domains $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset D$ forming some admissible perturbation and to approximate the original problem by the corresponding perturbed problems [12].

2. Notation and Preliminaries

Throughout the paper D and Ω are bounded open subsets of $\mathbb{R}^N$, $N \geqslant 1$ and $\Omega \subset\subset D$. Let χ_Ω be the characteristic function of the set Ω and let $\mathcal{L}^N(\Omega)$ be the N-dimensional Lebesgue measure of Ω. The space $\mathcal{D}'(\Omega)$ of distributions in Ω is the dual of the space $C_0^\infty(\Omega)$. For real numbers $2 \leqslant p < +\infty$, and $1 < q < +\infty$ such that $1/p+1/q = 1$, the space $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$ with respect to the norm

$$\|y\|_{W_0^{1,p}\Omega} = \left(\int_\Omega \sum_{k=1}^N \left|\frac{\partial y}{\partial x_i}\right|^p dx + \int_\Omega |y|^p\, dx\right)^{1/p}, \ \forall\, y \in W_0^{1,p}(\Omega), \tag{2.1}$$

while $W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$.

For any vector field $v \in L^q(\Omega;\mathbb{R}^N)$, the divergence is an element of the space $W^{-1,\,q}(\Omega)$ defined by the formula

$$\langle \operatorname{div} v, \varphi\rangle_{W_0^{1,p}(\Omega)} = -\int_\Omega (v, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega), \tag{2.2}$$

where $\langle\cdot,\cdot\rangle_{W_0^{1,p}(\Omega)}$ denotes the duality pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$, and $(\cdot,\cdot)_{\mathbb{R}^N}$ denotes the scalar product of two vectors in $\mathbb{R}^N$. A vector field $\mathbf{v}$ is said to be solenoidal, if $\operatorname{div}\mathbf{v} = 0$.

Weak Compactness Criterion in $L^1(\Omega)$. Throughout the paper we will often use the concepts of the weak and strong convergence in $L^1(\Omega)$. Let $\{a_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $L^1(\Omega)$. We recall that $\{a_\varepsilon\}_{\varepsilon>0}$ is called equi-integrable if for any $\delta > 0$ there is $\tau = \tau(\delta)$ such that $\int_S |a_\varepsilon|\, dx < \delta$ for every $\varepsilon > 0$ and every measurable subset $S \subset \Omega$ of Lebesgue measure $|S| < \tau$. Then the following assertions are equivalent: (i) A sequence $\{a_\varepsilon\}_{\varepsilon>0}$ is weakly compact in $L^1(\Omega)$. (ii) The sequence $\{a_\varepsilon\}_{\varepsilon>0}$ is equi-integrable. (iii) Given $\delta > 0$ there exists $\lambda = \lambda(\delta)$ such that $\sup_{\varepsilon>0}\int_{\{|a_\varepsilon|>\lambda\}} |a_\varepsilon|\, dx < \delta$.

Theorem 2.1 (Lebesgue's Theorem)**.** *If a bounded sequence* $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega)$ *is equi-integrable and* $a_\varepsilon \to a$ *almost everywhere on* Ω*, then* $a_\varepsilon \to a$ *in* $L^1(\Omega)$.

Functions with bounded variations. Let $f : \Omega \to \mathbb{R}$ be a function of $L^1(\Omega)$. Define

$$\begin{aligned} TV(f) &:= \int_\Omega |Df| \\ &= \sup\Big\{ \int_\Omega f(\nabla,\varphi)_{\mathbb{R}^N}\, dx \ :\ \varphi \in C_0^1(\Omega;\mathbb{R}^N),\ |\varphi(x)| \leqslant 1 \,\text{for}\, x \in \Omega\Big\}, \end{aligned}$$

where $(\nabla,\varphi)_{\mathbb{R}^N} = \sum_{i=1}^N \frac{\partial\varphi_i}{\partial x_i}$.

According to the Radon-Nikodym theorem, if $TV(f) < +\infty$ then the distribution Df is a measure and there exist a vector-valued function $\nabla f \in L^1(\Omega;\mathbb{R}^N)$ and a measure $D_s f$, singular with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N\lfloor\Omega$ restricted to Ω, such that $Df = \nabla f \mathcal{L}^N\lfloor\Omega + D_s f$.

Definition 2.1. A function $f \in L^1(\Omega)$ is said to have a bounded variation in Ω if $TV(f) < +\infty$. By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ with bounded variation, i.e. $BV(\Omega) = \left\{f \in L^1(\Omega)\ :\ TV(f) < +\infty\right\}$.

Under the norm $\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + TV(f)$, $BV(\Omega)$ is a Banach space. For our further analysis, we need the following properties of BV-functions (see [13]):

Proposition 2.1. (i) Let $\{f_k\}_{k=1}^\infty$ be a sequence in $BV(\Omega)$ strongly converging to some f in $L^1(\Omega)$ and satisfying condition $\sup_{k\in\mathbb{N}} TV(f_k) < +\infty$. Then

$$f \in BV(\Omega) \quad \text{and} \quad TV(f) \leqslant \liminf_{k\to\infty} TV(f_k);$$

(ii) for every $f \in BV(\Omega)\cap L^r(\Omega)$, $r \in [1, +\infty)$, there exists a sequence $\{f_k\}_{k=1}^\infty \subset C^\infty(\Omega)$ such that

$$\lim_{k\to\infty} \int_\Omega |f - f_k|^r \, dx = 0 \quad \text{and} \quad \lim_{k\to\infty} TV(f_k) = TV(f);$$

(iii) for every bounded sequence $\{f_k\}_{k=1}^\infty \subset BV(\Omega)$ there exists a subsequence, still denoted by f_k, and a function $f \in BV(\Omega)$ such that $f_k \to f$ in $L^1(\Omega)$.

Monotone operators. Let α and β be constants such that $0 < \alpha \leqslant \beta < +\infty$. We define $M_p^{\alpha,\beta}(D)$ as the set of all square symmetric matrices $\mathcal{U}(x) = [a_{i\,j}(x)]_{1\leqslant i,j\leqslant N}$ in $L^\infty(D; \mathbb{R}^{N\times N})$ such that the following conditions of growth, monotonicity, and strong coercivity are fulfilled:

$$|a_{ij}(x)| \leqslant \beta \quad \text{a.e. in} \ \ D, \ \forall\, i, j \in \{1, \dots, N\}, \tag{2.3}$$

$$\big(\mathcal{U}(x)([\zeta^{p-2}]\zeta - [\eta^{p-2}]\eta), \zeta - \eta\big)_{\mathbb{R}^N} \geqslant 0 \quad \text{a.e. in} \ \ D, \ \forall\, \zeta, \eta \in \mathbb{R}^N, \tag{2.4}$$

$$\big(\mathcal{U}(x)[\zeta^{p-2}]\zeta, \zeta\big)_{\mathbb{R}^N} = \sum_{i,j=1}^N a_{i\,j}(x)|\zeta_j|^{p-2}\,\zeta_j\,\zeta_i \geqslant \alpha\,|\zeta|_p^p \quad \text{a.e in} \ \ D, \tag{2.5}$$

where $|\eta|_p = \left(\sum\limits_{k=1}^N |\eta_k|^p\right)^{1/p}$ is the Hölder norm of $\eta \in \mathbb{R}^N$ and

$$[\eta^{p-2}] = \operatorname{diag}\{|\eta_1|^{p-2}, |\eta_2|^{p-2}, \dots, |\eta_N|^{p-2}\}, \quad \forall \eta \in \mathbb{R}^N. \tag{2.6}$$

Remark 2.1. It is easy to see that $M_p^{\alpha,\beta}(D)$ is a nonempty subset of $L^\infty(D; \mathbb{R}^{N\times N})$. As the particular representatives of the set $M_p^{\alpha,\beta}(D)$ we can take diagonal matrices of the form (see [10]), $\mathcal{U}(x) = \operatorname{diag}\{\delta_1(x), \delta_2(x), \dots, \delta_N(x)\}$, where $\alpha \leqslant \delta_i(x) \leqslant \beta$ a.e. in D $\forall\, i \in \{1, \dots, N\}$.

Let us consider a nonlinear operator $A : M_p^{\alpha,\beta}(D) \times W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ defined as

$$A(\mathcal{U}, y) = -\operatorname{div}\big(\mathcal{U}(x)[(\nabla y)^{p-2}]\nabla y\big) + |y|^{p-2}y,$$

or via the paring

$$\langle A(\mathcal{U}, y), \varphi\rangle_{W_0^{1,p}(\Omega)} = \sum_{i,j=1}^N \int_\Omega \left(a_{ij}(x)\left|\frac{\partial y}{\partial x_j}\right|^{p-2} \frac{\partial y}{\partial x_j}\right) \frac{\partial \varphi}{\partial x_i}\, dx + \int_\Omega |y|^{p-2} y\,\varphi\, dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega).$$

In view of properties (2.3)–(2.5), for every fixed matrix $\mathcal{U} \in M_p^{\alpha,\beta}(D)$, the operator $A(\mathcal{U}, \cdot)$ turns out to be coercive, strongly monotone and demi-continuous in the following sense: $y_k \to y_0$ strongly in $W_0^{1,p}(\Omega)$ implies that $A(\mathcal{U}, y_k) \rightharpoonup A(\mathcal{U}, y_0)$ weakly in $W^{-1,q}(\Omega)$ (see [15]). Then by well-known existence results for nonlinear elliptic equations with strictly monotone demi-continuous coercive operators (see [15, 26]), the nonlinear Dirichlet boundary value problem

$$A(\mathcal{U}, y) = f \quad \text{in} \quad \Omega, \qquad y \in W_0^{1,p}(\Omega), \tag{2.7}$$

admits a unique weak solution in $W_0^{1,p}(\Omega)$ for every fixed matrix $\mathcal{U} \in M_p^{\alpha,\beta}(D)$ and every distribution $f \in W^{-1,q}(D)$. Let us recall that a function y is the weak solution of (2.7) if

$$y \in W_0^{1,p}(\Omega), \tag{2.8}$$

$$\int_\Omega \big(\mathcal{U}(x)[(\nabla y)^{p-2}]\nabla y, \nabla \varphi\big)_{\mathbb{R}^N}\, dx + \int_\Omega |y|^{p-2} y \varphi \, dx = \int_\Omega f \varphi \, dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega). \tag{2.9}$$

System of nonlinear operator equations with an equation of Hammerstein type. Let Y and Z be Banach spaces, let $Y_0 \subset Y$ be an arbitrary bounded set, and let Z^* be the dual space to Z. Let V be a dual space to some Banach space B and $V_0 \subset V$ be a bounded subset. To begin with we recall some useful properties of non-linear operators, concerning the solvability problem for Hammerstein type equations and systems.

Definition 2.2. We say that the operator $G : D(G) \subset Z \to Z^*$ is radially continuous if for any $z_1, z_2 \in X$ there exist $\varepsilon > 0$ such that $z_1 + \tau z_2 \in D(G)$ for all $\tau \in [0, \varepsilon]$ and the real-valued function $[0, \varepsilon] \ni \tau \to \langle G(z_1 + \tau z_2), z_2 \rangle_Z$ is continuous.

Definition 2.3. An operator $G : V \times Y \times Z \to Z^*$ is said to have a uniformly semi-bounded variation (u.s.b.v.) if for any bounded set $V_0 \times Y_0 \subset V \times Y$ and any elements $z_1, z_2 \in D(G)$ such that $\|z_i\|_Z \leq R$, $i = 1, 2$, the following inequality

$$\langle G(v, y, z_1) - G(v, y, z_2), z_1 - z_2 \rangle_Z \geqslant - \inf_{(v,y) \in V_0 \times Y_0} C_{v,y}(R; |||z_1 - z_2|||_Z) \tag{2.10}$$

holds true provided the function $C_{v,y} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous for each pair $(v, y) \in V_0 \times Y_0$, and $\frac{1}{t} C_{v,y}(r, t) \to 0$ as $t \to 0$, $\forall r > 0$. Here, $||| \cdot |||_Z$ is a seminorm on Z such that $||| \cdot |||_Z$ is compact with respect to the norm $\| \cdot \|_Z$.

It is worth to mention here that if $C_{v,y}(\rho, r) \equiv 0$, then (2.10) implies the monotonicity property for the operator G with respect to the third argument.

Remark 2.2. Each operator $G : V \times Y \times Z \to Z^*$ with u.s.b.v. possesses the following property (see for comparison Remark 1.1.2 in [1]): if a set $K \subset Z$ is such that $\|z\|_Z \leqslant k_1$ and $\langle G(v, y, z), z \rangle_Z \leqslant k_2$ for all $z \in K$ and $(v, y) \in V_0 \times Y_0$, then there exists a constant $C > 0$ such that $\|G(v, y, z)\|_{Z^*} \leqslant C$, $\forall\, z \in K$ and $\forall (v, y) \in V_0 \times Y_0$.

Let $B : Z^* \to Z$ and $F : V \times Y \times Z \to Z^*$ be given operators such that the mapping $Z^* \ni z^* \mapsto B(z^*) \in Z$ is linear. Let $g \in Z$ be a given distribution. Then a typical Hammerstein operator equation can be represented as follows

$$z + BF(v, y, z) = g, \tag{2.11}$$

The following existence result is well-known (see [1, Theorem 1.2.1]).

Theorem 2.2. *Let $B : Z^* \to Z$ be a linear continuous positive operator such that it has the right inverse operator $B_r^{-1} : Z \to Z^*$. Let $F : V \times Y \times Z \to Z^*$ be an operator with u.s.b.v such that $F(v, y, \cdot) : Z \to Z^*$ is radially continuous for each pair $(v, y) \in V_0 \times Y_0$ and the following inequality holds true*

$$\langle F(v, y, z) - B_r^{-1} g, z\rangle_Z \geqslant 0 \quad \text{if only } \|z\|_Z > \lambda > 0,\ \lambda = const.$$

Then the set

$$\mathcal{H}(v, y) = \{z \in Z : \ z + BF(v, y, z) = g \text{ in the sense of distributions }\}$$

is non-empty and weakly compact for every fixed pair $(v, y) \in V_0 \times Y_0$ and $g \in Z$.

Definition 2.4. We say that

($\mathfrak{M}$) the operator $F : V \times Y \times Z \to Z^*$ possesses the $\mathfrak{M}$-property if for any sequences $\{v_k\}_{k\in\mathbb{N}} \subset V$, $\{y_k\}_{k\in\mathbb{N}} \subset Y$ and $\{z_k\}_{k\in\mathbb{N}} \subset Z$ such that $v_k \to v$ strongly in V, $y_k \to y$ strongly in Y and $z_k \to z$ weakly in Z as $k \to \infty$, the condition

$$\lim_{k\to\infty} \langle F(v_k, y_k, z_k), z_k\rangle_Z = \langle F(v, y, z), z\rangle_Z \tag{2.12}$$

implies that $z_k \to z$ strongly in Z.

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$$\liminf_{k\to\infty} \langle F(v_k, y_k, z_k), z_k\rangle_Z \geqslant \langle F(v, y, z), z\rangle_Z \tag{2.13}$$

holds true.

In what follows, we set $V = L^\infty(D)$, $Y = W_0^{1,p}(\Omega)$, $Z = L^p(\Omega)$, and $Z^* = L^q(\Omega)$.

2.1. Capacity

There are many ways to define the Sobolev capacity. We use the notion of local p-capacity which can be defined in the following way:

Definition 2.5. For a compact set K contained in an arbitrary ball B, capacity of K in B, denoted by $C_p(K,B)$, is defined as follows

$$C_p(K,B) = \inf\left\{ \int_B |D\varphi|^p \,dx, \quad \forall\, \varphi \in C_0^\infty(B),\ \varphi \geqslant 1 \ \text{ on } \ K \right\}.$$

For open sets contained in B the capacity is defined by an interior approximating procedure by compact sets (see [16]), and for arbitrary sets by an exterior approximating procedure by open sets.

It is said that a property holds p-quasi everywhere (abbreviated as p-q.e.) if it holds outside a set of p-capacity zero. It is said that a property holds almost everywhere (abbreviated as a.e.) if it holds outside a set of Lebesgue measure zero.

A function y is called p-quasi–continuous if for any $\delta > 0$ there exists an open set A_δ such that $C_p(A_\delta, B) < \delta$ and y is continuous in $D \setminus A_\delta$. We recall that any function $y \in W^{1,p}(D)$ has a unique (up to a set of p-capacity zero) p-quasi continuous representative. Let us recall the following results (see [2,16]):

Theorem 2.3. *If $y \in W^{1,p}(\mathbb{R}^N)$, then $y|_\Omega \in W_0^{1,p}(\Omega)$ if and only if $y = 0$ p-q.e. on Ω^c for a p-quasi-continuous representative.*

Theorem 2.4. *Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $y \in W^{1,p}(\Omega)$. If $y = 0$ a.e. in Ω, then $y = 0$ p-q.e. in Ω.*

For these and other properties on quasi-continuous representatives, the reader is referred to [2,13,16,27].

2.2. Convergence of sets

In order to speak about "domain perturbation", we have to prescribe a topology on the space of open subsets of D. To do this, for the family of all open subsets of D, we define the Hausdorff complementary topology, denoted by H^c, given by the metric:

$$d_{H^c}(\Omega_1, \Omega_2) = \sup_{x \in \mathbb{R}^N} \left| d(x, \Omega_1^c) - d(x, \Omega_2^c) \right|,$$

where Ω_i^c are the complements of Ω_i in $\mathbb{R}^N$.

Definition 2.6. We say that a sequence $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of open subsets of D converges to an open set $\Omega \subseteq D$ in H^c-topology, if $d_{H^c}(\Omega_\varepsilon, \Omega)$ converges to 0 as $\varepsilon \to 0$.

The H^c-topology has some good properties, namely the space of open subsets of D is compact with respect to H^c-convergence, and if $\Omega_\varepsilon \stackrel{H^c}{\longrightarrow} \Omega$, then for any compact $K \subset\subset \Omega$ we have $K \subset\subset \Omega_\varepsilon$ for ε small enough. Moreover, a sequence of open sets $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset D$ H^c-converges to an open set Ω, if and only if the sequence of complements $\{\Omega_\varepsilon^c\}_{\varepsilon>0}$ converges to Ω^c in the sense of Kuratowski. We recall here that a sequence $\{C_\varepsilon\}_{\varepsilon>0}$ of closed subsets of $\mathbb{R}^N$ is said to be convergent to a closed set C in the sense of Kuratowski if the following two properties hold:

(K_1) for every $x \in C$, there exists a sequence $\{x_\varepsilon \in C_\varepsilon\}_{\varepsilon>0}$ such that $x_\varepsilon \to x$ as $\varepsilon \to 0$;

(K_2) if $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is a sequence of indices converging to zero, $\{x_k\}_{k\in\mathbb{N}}$ is a sequence such that $x_k \in C_{\varepsilon_k}$ for every $k \in \mathbb{N}$, and x_k converges to some $x \in \mathbb{R}^N$, then $x \in C$.

For these and other properties on H^c-topology, we refer to [14].

It is well known that in the case when $p > N$, the H^c-convergence of open sets $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset D$ is equivalent to the convergence in the sense of Mosco of the associated Sobolev spaces.

Definition 2.7. We say a sequence of spaces $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ converges in the sense of Mosco to $W_0^{1,p}(\Omega)$ (see for comparison [23]) if the following conditions are satisfied:

(M_1) for every $y \in W_0^{1,p}(\Omega)$ there exists a sequence $\left\{y_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ such that $\widetilde{y}_\varepsilon \to \widetilde{y}$ strongly in $W^{1,p}(\mathbb{R}^N)$;

(M_2) if $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is a sequence converging to 0 and $\left\{y_k \in W_0^{1,p}(\Omega_{\varepsilon_k})\right\}_{k\in\mathbb{N}}$ is a sequence such that $\widetilde{y}_k \to \psi$ weakly in $W^{1,p}(\mathbb{R}^N)$, then there exists a function $y \in W_0^{1,p}(\Omega)$ such that $y = \psi|_\Omega$.

Hereinafter we denote by $\widetilde{y}_\varepsilon$ (respect. $\widetilde{y}$) the zero-extension to $\mathbb{R}^N$ of a function defined on Ω_ε (respect. on Ω), that is, $\widetilde{y}_\varepsilon = \widetilde{y}_\varepsilon\chi_{\Omega_\varepsilon}$ and $\widetilde{y} = \widetilde{y}\chi_\Omega$.

Following Bucur & Trebeschi (see [4]), we have the following result.

Theorem 2.5. *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a sequence of open subsets of D such that $\Omega_\varepsilon \xrightarrow{H^c} \Omega$ and $\Omega_\varepsilon \in \mathcal{W}_w(D)$ for every $\varepsilon > 0$, with the class $\mathcal{W}_w(D)$ defined as*

$$\begin{aligned}\mathcal{W}_w(D) = \{\Omega \subseteq D \ :\ \forall\, x \in \partial\Omega, \forall\, 0 < r < R < 1;& \\ \int_r^R \left(\frac{C_p(\Omega^c \cap \overline{B(x,t)}; B(x,2t))}{C_p(\overline{B(x,t)}; B(x,2t))}\right)^{\frac{1}{p-1}} \frac{dt}{t} \geqslant w(r,R,x)&\Bigg\},\end{aligned} \tag{2.14}$$

where $B(x,t)$ is the ball of radius t centered at x, and the function

$$w : (0,1) \times (0,1) \times D \to \mathbb{R}^+$$

is such that

1. $\lim_{r\to 0} w(r,R,x) = +\infty$, *locally uniformly on $x \in D$;*

2. *w is a lower semicontinuous function in the third argument.*

Then $\Omega \in \mathcal{W}_w(D)$ and the sequence of Sobolev spaces $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ converges in the sense of Mosco to $W_0^{1,p}(\Omega)$.

Theorem 2.6. *Let $N \geqslant p > N-1$ and let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a sequence of open subsets of D such that $\Omega_\varepsilon \xrightarrow{H^c} \Omega$ and $\Omega_\varepsilon \in \mathcal{O}_l(D)$ for every $\varepsilon > 0$, where the class $\mathcal{O}_l(D)$ is defined as follows*

$$\mathcal{O}_l(D) = \{\Omega \subseteq D \ : \ \sharp\Omega^c \leqslant l\} \tag{2.15}$$

(here by $\sharp$ one denotes the number of connected components). Then $\Omega \in \mathcal{O}_l(D)$ and the sequence of Sobolev spaces $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ converges in the sense of Mosco to $W_0^{1,p}(\Omega)$.

In the meantime, the perturbation in H^c-topology (without some additional assumptions) may be very irregular. It means that the continuity of the mapping $\Omega \mapsto y_\Omega$, which associates to every Ω the corresponding solution y_Ω of a Dirichlet boundary problem (2.8)–(2.9), may fail (see, for instance, [7]). In view of this, we introduce one more concept of the set convergence. Following Dancer [8] (see also [9]), we say that

Definition 2.8. A sequence $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of open subsets of D topologically converges to an open set $\Omega \subseteq D$ (in symbols $\Omega_\varepsilon \xrightarrow{\text{top}} \Omega$) if there exists a compact set $K_0 \subset \Omega$ of p-capacity zero ($C_p(K_0, D) = 0$) and a compact set $K_1 \subset \mathbb{R}^N$ of Lebesgue measure zero such that

(D_1) $\Omega' \subset\subset \Omega \setminus K_0$ implies that $\Omega' \subset\subset \Omega_\varepsilon$ for ε small enough;

(D_2) for any open set U with $\overline{\Omega} \cup K_1 \subset U$, we have $\Omega_\varepsilon \subset U$ for ε small enough.

Note that without supplementary regularity assumptions on the sets, there is no connection between this type of set convergence and the set convergence in the Hausdorff complementary topology. Moreover, the topological set convergence allows certain parts of the subsets Ω_ε degenerating and being deleted in the limit. For instance, assume that Ω consists of two disjoint balls, and Ω_ε is a dumbbell with a small hole on each side. Shrinking the holes and the handle, we can approximate the set Ω by sets Ω_ε in the sense of Definition 2.8 as shown in Figure 1. It is obvious that in this case $d_{H^c}(\Omega_\varepsilon, \Omega)$ does not converge to 0 as

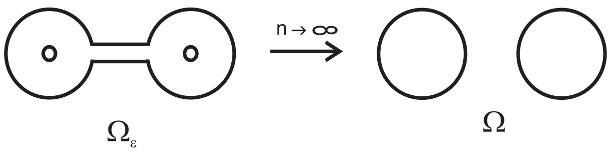

Fig. 1: Example of the set convergence in the sense of Definition 2.8

$\varepsilon \to 0$. However, as an estimate of an "approximation" of Ω by elements of the above sequence $\Omega_\varepsilon \xrightarrow{\text{top}} \Omega$, we can take the Lebesgue measure of the symmetric set difference $\Omega_\varepsilon \triangle \Omega$, that is, $\mu(\Omega, \Omega_\varepsilon) = \mathcal{L}^N(\Omega \setminus \Omega_\varepsilon \cup \Omega_\varepsilon \setminus \Omega)$. It should be noted that in this case the distance μ coincides with the well-known Ekeland metric in $L^\infty(D)$ applied to characteristic functions:

$$d_E(\chi_\Omega, \chi_{\Omega_\varepsilon}) = \mathcal{L}^N \left\{x \in D \ : \ \chi_\Omega(x) \neq \chi_{\Omega_\varepsilon}(x)\right\} = \mu(\Omega, \Omega_\varepsilon).$$

As an example of subsets which are H^c-convergent but have no limit in the sense of Definition 2.8, let us consider the sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ containing an oscillating crack with vanishing amplitude ε (see Figure 2).

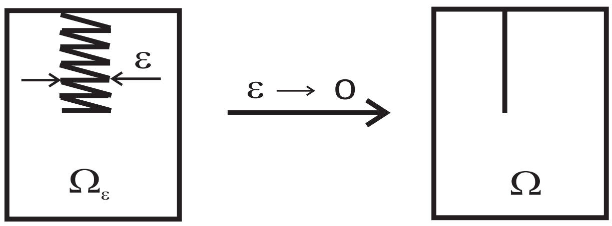

Fig. 2: The p-unstable sets which are compact with respect to the H^c-topology

3. Setting of the Optimal Control Problem and Existence Result

Let ξ_1, ξ_2 be given functions of $L^\infty(D)$ such that $0 \leqslant \xi_1(x) \leqslant \xi_2(x)$ a.e. in D. Let $\{Q_1, \dots, Q_N\}$ be a collection of nonempty compact convex subsets of $W^{-1,q}(D)$. To define the class of admissible controls, we introduce two sets

$$U_b = \left\{ \mathcal{U} = [a_{ij}] \in M_p^{\alpha,\beta}(D) \middle| \, \xi_1(x) \le a_{ij}(x) \le \xi_2(x) \, \text{a.e. in} \, D, \ \forall\, i,j = 1, \dots, N \right\}, \tag{3.1}$$

$$U_{sol} = \left\{ \mathcal{U} = [u_1, \dots, u_N] \in M_p^{\alpha,\beta}(D) \middle| \, \operatorname{div} u_i \in Q_i, \ \forall\, i = 1, \dots, N \right\}, \tag{3.2}$$

assuming that the intersection $U_b \cap U_{sol} \subset L^\infty(D; \mathbb{R}^{N\times N})$ is nonempty. We say that a matrix $\mathcal{U} = [a_{i\,j}]$ is of solenoidal type if $\mathcal{U} \in U_{ad} := U_b \cap U_{sol}$

Definition 3.1. We say that a pair $(\mathcal{U}, v)$ is an admissible control if

$$(\mathcal{U}, v) \in U_{ad} \times V_{ad},$$

where $V_{ad} \subset L^\infty(D)$ is an appropriate bounded subset.

Remark 3.1. As it was shown in [10] the set U_{ad} is compact with respect to the weak-$*$ topology of the space $L^\infty(D; \mathbb{R}^{N\times N})$ as well as V_{ad} is obviously weakly-$*$ compact in $L^\infty(D)$.

Let us consider the following optimal control problem:

$$\text{Minimize } \Big\{ I_\Omega(\mathcal{U}, v, y, z) = \int_\Omega |z(x) - z_d(x)|^p \, dx \Big\}, \tag{3.3}$$

subject to the constraints

$$\int_\Omega \big(\mathcal{U}[(\nabla y)^{p-2}]\nabla y, \nabla \varphi\big)_{\mathbb{R}^N} \, dx + \int_\Omega |y|^{p-2} y \varphi \, dx = \langle f, \varphi \rangle_{W_0^{1,p}(\Omega)}, \; \forall \varphi \in W_0^{1,p}(\Omega), \tag{3.4}$$

$$\mathcal{U} \in U_{ad}, \quad y \in W_0^{1,p}(\Omega), \tag{3.5}$$

$$\int_\Omega z \, \phi \, dx + \int_\Omega BF(v, y, z) \, \phi \, dx = \int_\Omega g \, \phi \, dx, \tag{3.6}$$

$$v \in V_{ad}. \tag{3.7}$$

where $f \in W^{-1,q}(D)$, $g \in L^p(D)$, and $z_d \in L^p(D)$ are given distributions.

Hereinafter, $\Xi_{sol} \subset L^\infty(D; \mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega)$ denotes the set of all admissible quadruples to the optimal control problem (3.3)–(3.7). Let τ be the topology on the set $L^\infty(D; \mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega)$ which we define as a product of the weak-$*$ topology of $L^\infty(D; \mathbb{R}^{N\times N}) \times L^\infty(D)$, the weak topology of $W_0^{1,p}(\Omega)$, and the weak topology of $L^p(\Omega)$.

Further we use the following result (see [10, 19]).

Proposition 3.1. For each $\mathcal{U} \in U_{ad}$ and every $f \in W^{-1,q}(D)$, a weak solution y to variational problem (3.4)–(3.5) satisfies the estimate

$$\|y\|^p_{W_0^{1,p}(\Omega)} \leqslant C \|f\|^q_{W^{-1,q}(D)}, \tag{3.8}$$

where C is a constant depending on p and α only.

Proposition 3.2. Let $B : L^q(\Omega) \to L^p(\Omega)$ and $F : L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega) \to L^q(\Omega)$ be operators satisfying all conditions of Theorem 2.2. Then the set

$$\Xi_{sol} = \big\{ (\mathcal{U}, v, y, z) \in L^\infty(D; \mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega) : \\ A(\mathcal{U}, y) = f, \; z + BF(v, y, z) = g) \big\}$$

is nonempty for every $f \in W^{-1,q}(D)$ and $g \in L^p(D)$.

Proof. Let $(\mathcal{U}, v) \in U_{ad} \times V_{ad}$ be an arbitrary admissible control. Then for a given $f \in W^{-1,q}(D)$, the Dirichlet boundary problem (3.4)–(3.5) admits a unique solution $y_{\mathcal{U}} = y(\mathcal{U}, f) \in W_0^{1,p}$ for which the estimate (3.8) holds true. It remains to remark that the corresponding Hammerstein equation

$$z + BF(v, y_{\mathcal{U}}, z) = g \tag{3.9}$$

has a nonempty set of solutions $\mathcal{H}(v, y_{\mathcal{U}})$ for every $g \in L^p(D)$ by Theorem 2.2. $\square$

Theorem 3.1. *Assume the following conditions hold:*

- *the operators* $B : L^q(\Omega) \to L^p(\Omega)$ *and* $F : L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega) \to L^q(\Omega)$ *satisfy conditions of Theorem 2.2;*
- *the operator* $F(\cdot,\cdot,z) : L^\infty(D) \times W_0^{1,p}(\Omega) \to L^q(\Omega)$ *is compact in the following sense: if* $v_k \to v$ *weakly-$*$ in* $L^\infty(D)$ *and* $y_k \to y_0$ *weakly in* $W_0^{1,p}(\Omega)$*, then* $F(v_k, y_k, z) \to F(v_0, y_0, z)$ *strongly in* $L^q(\Omega)$*.*

Then for every $f \in W^{-1,q}(D)$ *and* $g \in L^p(D)$*, the set* Ξ_{sol} *is sequentially* τ*-closed, i.e. if a sequence* $\{(\mathcal{U}_k, v_k, y_k, z_k) \in \Xi_{sol}\}_{k\in\mathbb{N}}$ *is such that* $\mathcal{U}_k \to \mathcal{U}_0$ *weakly-$*$ in* $L^\infty(\Omega;\mathbb{R}^{N\times N})$*,* $v_k \to v$ *weakly-$*$ in* $L^\infty(D)$*,* $y_k = y(\mathcal{U}_k) \to y_0$ *weakly in* $W_0^{1,p}(\Omega)$*, and* $z_k = z(v_k, y_k) \to z_0$ *weakly in* $L^p(\Omega)$*, then* $(\mathcal{U}_0, v_0) \in U_{ad} \times V_{ad}$*,* $y_0 = y(\mathcal{U}_0)$*,* $z_0 \in \mathcal{H}(v_0, y_0)$*, and, therefore,* $(\mathcal{U}_0, v_0, y_0, z_0) \in \Xi_{sol}$*.*

Proof. Let $\{(\mathcal{U}_k, v_k, y_k, z_k)\}_{k\in\mathbb{N}} \subset \Xi_{sol}$ be any τ-convergent sequence of admissible quadruples to the optimal control problem (3.3)–(3.7), and let $(\mathcal{U}_0, v_0, y_0, z_0)$ be its τ-limit. Since the controls $\{\mathcal{U}_k\}_{k\in\mathbb{N}}$ belong to the set of solenoidal matrices U_{sol} (see (3.2)), it follows from [18, 20] that $\mathcal{U}_0 \in U_{ad}$ and $y_0 = y(\mathcal{U}_0)$. It remains to show that $z_0 \in \mathcal{H}(v_0, y_0)$. To this end, we have to pass to the limit in equation

$$z_k + BF(v_k, y_k, z_k) = g \tag{3.10}$$

as $k \to \infty$ and get the limit triplet (v_0, y_0, z_0) is related by the equation $z_0 + BF(v_0, y_0, z_0) = g$. With that in mind, let us rewrite equation (3.10) in the following way

$$B^* w_k + BF(v_k, y_k, B^* w_k) = g,$$

where $w_k \in L^q(\Omega)$, $B^* : L^q(\Omega) \to L^p(\Omega)$ is the conjugate operator for B, i.e. $\langle B\nu, w\rangle_{L^q(\Omega)} = \langle B^* w, \nu\rangle_{L^q(\Omega)}$ and $B^* w_k = z_k$. Then, for every $k \in \mathbb{N}$, we have the equality

$$\langle B^* w_k, w_k\rangle_{L^p(\Omega)} + \langle F(v_k, y_k, B^* w_k), B^* w_k\rangle_{L^p(\Omega)} = \langle g, w_k\rangle_{L^p(\Omega)}. \tag{3.11}$$

Taking into account the transformation

$$\langle g, w_k\rangle_{L^p(\Omega)} = \langle BB_r^{-1} g, w_k\rangle_{L^p(\Omega)} = \langle B_r^{-1} g, B^* w_k\rangle_{L^p(\Omega)},$$

we obtain

$$\langle w_k, Bw_k\rangle_{L^p(\Omega)} + \langle F(v_k, y_k, B^* w_k) - B_r^{-1} g, B^* w_k\rangle_{L^p(\Omega)} = 0. \tag{3.12}$$

The first term in (3.12) is strictly positive for every $w_k \neq 0$, hence, the second one must be negative. In view of the initial assumptions, namely,

$$\langle F(v, y, x) - B_r^{-1} g, x\rangle_{L^p(\Omega)} \geqslant 0 \quad \text{if only } \|x\|_{L^p(\Omega)} > \lambda,$$

we conclude that

$$\|B^* w_k\|_{L^p(\Omega)} = \|z_k\|_{L^p(\Omega)} \leqslant \lambda. \tag{3.13}$$

Since the linear positive operator B^* cannot map unbounded sets into bounded ones, it follows that $\|w_k\|_{L^q(\Omega)} \leqslant \lambda_1$. As a result, see (3.11), we have

$$\langle F(v_k, y_k, B^* w_k), B^* w_k\rangle_{L^p(\Omega)} = -\langle B w_k, w_k\rangle_{L^p(\Omega)} + \langle g, w_k\rangle_{L^p(\Omega)}, \tag{3.14}$$

and, therefore, $\langle F(v_k, y_k, B^* w_k), B^* w_k\rangle_{L^p(\Omega)} \leqslant c_1$. Indeed, all terms in the right-hand side of (3.14) are bounded provided the sequence $\{w_k\}_{k\in\mathbb{N}} \subset L^q(\Omega)$ is bounded and operator B is linear and continuous. Hence, in view of Remark 2.2, we get

$$\|F(v_k, y_k, B^* w_k)\|_{L^q(\Omega)} = \|F(v_k, y_k, z_k)\|_{L^q(\Omega)} \leqslant c_2 \quad \text{as} \quad \|z_k\|_{L^p(\Omega)} \leqslant \lambda.$$

Since the right-hand side of (3.14) does not depend on v_k and y_k, it follows that the constant $c_2 > 0$ does not depend on v_k and y_k either.

Taking these arguments into account, we may suppose that up to a subsequence we have the weak convergence $F(v_k, y_k, z_k) \to \nu_0$ in $L^q(\Omega)$. As a result, passing to the limit in (3.10), by continuity of B, we finally get

$$z_0 + B\nu_0 = g. \tag{3.15}$$

It remains to show that $\nu_0 = F(v_0, y_0, z_0)$. Let us take an arbitrary element $z \in L^p(\Omega)$ such that $\|z\|_{L^p(\Omega)} \leqslant \lambda$. Using the fact that F is an operator with u.s.b.v., we have

$$\langle F(v_k, y_k, z) - F(v_k, y_k, z_k), z - z_k\rangle_{L^p(\Omega)} \geqslant - \inf_{(v,y)\in V_{ad}\times Y_0} C_{v,y}(\lambda; |||z - z_k|||_{L^p(\Omega)}),$$

where $Y_0 = \{y \in W_0^{1,p}(\Omega) :\ y \text{ satisfies } (3.8)\}$, or, after transformation,

$$\begin{aligned} &\langle F(v_k, y_k, z), z - z_k\rangle_{L^p(\Omega)} - \langle F(v_k, y_k, z_k), z\rangle_{L^p(\Omega)} \\ &\qquad \geqslant \langle F(v_k, y_k, z_k), -z_k\rangle_{L^p(\Omega)} - \inf_{(v,y)\in V_{ad}\times Y_0} C_{v,y}(\lambda; |||z - z_k|||_{L^p(\Omega)}). \end{aligned} \tag{3.16}$$

Since $-z_k = BF(v_k, y_k, z_k) - g$, it follows from (3.16) that

$$\begin{aligned} &\langle F(v_k, y_k, z), z - z_k\rangle_{L^p(\Omega)} - \langle F(v_k, y_k, z_k), z\rangle_{L^p(\Omega)} + \langle F(v_k, y_k, z_k), g\rangle_{L^p(\Omega)} \\ &\quad \geqslant \langle F(v_k, y_k, z_k), BF(v_k, y_k, z_k)\rangle_{L^p(\Omega)} - \inf_{(v,y)\in V_{ad}\times Y_0} C_{v,y}(\lambda; |||z - z_k|||_{L^p(\Omega)}). \end{aligned} \tag{3.17}$$

In the meantime, due to the weak convergence $F(z_k, y_k, z_k) \to \nu_0$ in $L^q(\Omega)$ as $k \to \infty$, we arrive at the following obvious properties

$$\liminf_{k\to\infty} \langle F(v_k, y_k, z_k), BF(v_k, y_k, z_k)\rangle_{L^p(\Omega)} \geqslant \langle \nu_0, B\nu_0\rangle_{L^p(\Omega)}, \tag{3.18}$$

$$\lim_{k\to\infty} \langle F(v_k, y_k, z_k), z\rangle_{L^p(\Omega)} = \langle \nu_0, z\rangle_{L^p(\Omega)}, \tag{3.19}$$

$$\lim_{k\to\infty} \langle F(v_k, y_k, z_k), g\rangle_{L^p(\Omega)} = \langle \nu_0, g\rangle_{L^p(\Omega)}. \tag{3.20}$$

Moreover, the continuity of the function $C_{v,y}$ with respect to the second argument and the compactness property of operator F, which means that $F(v_k, y_k, z) \to F(v_0, y_0, z)$ strongly in $L^q(\Omega)$, lead to the conclusion

$$\lim_{k\to\infty} C_{v,y}(\lambda; |||z - z_k|||_{L^p(\Omega)}) = C_{v,y}(\lambda; |||z - z_0|||_{L^p(\Omega)}), \quad \forall\, (v,\, y) \in V_{ad} \times Y_0, \tag{3.21}$$

$$\lim_{k\to\infty} \langle F(v_k, y_k, z), z - z_k\rangle_{L^p(\Omega)} = \langle F(v_0, y_0, z), z - z_0\rangle_{L^p(\Omega)}. \tag{3.22}$$

As a result, using the properties (3.18)–(3.22), we can pass to the limit in (3.17) as $k \to \infty$. One gets

$$\langle F(v_0, y_0, z), z - z_0\rangle_{L^p(\Omega)} - \langle \nu_0, z + B\nu_0 - g\rangle_{L^p(\Omega)} \geqslant - \inf_{(v,y)\in V_{ad}\times Y_0} C_{v,y}(\lambda; |||z - z_0|||_{L^p(\Omega)}). \tag{3.23}$$

Since $B\nu_0 - g = -z_0$ by (3.15), we can rewrite the inequality (3.23) as follows

$$\langle F(v_0, y_0, z) - \nu_0, z - z_0\rangle_{L^p(\Omega)} \geqslant - \inf_{(v,y)\in V_{ad}\times Y_0} C_{v,y}(\lambda; |||z - z_0|||_{L^p(\Omega)}).$$

It remains to note that the operator F is radially continuous for each pair $(v, y) \in V_{ad} \times Y_0$, and F is the operator with u.s.b.v. Therefore, the last relation implies that $F(v_0, y_0, z_0) = \nu_0$ (see [1, Theorem 1.1.2]) and, hence, equality (3.15) finally takes the form

$$z_0 + BF(v_0, y_0, z_0) = g. \tag{3.24}$$

Thus, $z_0 \in \mathcal{H}(v_0, y_0)$ and the triplet $(\mathcal{U}_0, v_0, y_0, z_0)$ is admissible for OCP (3.3)–(3.7). The proof is complete. $\square$

Remark 3.2. In fact, as immediately follows from the proof of Theorem 3.1, the set of admissible solutions Ξ to the problem (3.3)–(3.7) is sequentially τ-compact.

The next observation is important for our further analysis.

Corollary 3.1. *Assume that all preconditions of Theorem 3.1 hold true. Assume also that the operator $F : L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega) \to L^q(\Omega)$ possesses $(\mathfrak{M})$ and $(\mathfrak{A})$ properties. Let $\{v_k\}_{k\in\mathbb{N}}$ be a strongly convergent sequence in $L^\infty(D)$ and $\{y_k\}_{k\in\mathbb{N}}$ be a strongly convergent sequence in $W_0^{1,p}(\Omega)$. Then an arbitrary chosen sequence $\{z_k \in \mathcal{H}(v_k, y_k)\}_{k\in\mathbb{N}}$ is relatively compact with respect to the strong topology of $L^p(\Omega)$, i.e. there exists an element $z_0 \in \mathcal{H}(v_0, y_0)$ such that within a subsequence*

$$z_k \to z_0 \quad \textit{strongly in} \quad L^p(\Omega) \quad \textit{as} \quad k \to \infty.$$

Proof. Let $\{v_k\}_{k\in\mathbb{N}} \subset L^\infty(D)$ and $\{y_k\}_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ be given sequences, and let $v_0 \in L^\infty(D)$ and $y_0 \in W_0^{1,p}(\Omega)$ be their strong limits. Let $\{z_k \in \mathcal{H}(v_k, y_k)\}_{k\in\mathbb{N}}$ be an arbitrary sequence of corresponding solutions to the Hammerstein equation (3.6)–(3.7).

As follows from the proof of Theorem 3.1, the sequence $\{z_k \in \mathcal{H}(v_k, y_k)\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^p(\Omega)$ and, moreover, there exist a subsequence of $\{z_k\}_{k\in\mathbb{N}}$ still denoted by the same index and an element $z_0 \in L^p(\Omega)$ such that $z_k \to z_0$ weakly in $L^p(\Omega)$ and $z_0 \in \mathcal{H}(v_0, y_0)$. Our aim is to show that in this case $z_k \to z_0$ strongly in $L^p(\Omega)$. Indeed, as follows from (3.10) and (3.24), we have the following equalities

$$\begin{aligned}\langle F(v_k, y_k, z_k), z_k\rangle_{L^p(\Omega)} &+ \langle F(v_k, y_k, z_k), BF(v_k, y_k, z_k)\rangle_{L^p(\Omega)} \\ &= \langle F(v_k, y_k, z_k), g\rangle_{L^p(\Omega)}, \ \forall k \in \mathbb{N}, \quad (3.25)\end{aligned}$$

$$\begin{aligned}\langle F(v_0, y_0, z_0), z_0\rangle_{L^p(\Omega)} &+ \langle F(v_0, y_0, z_0), BF(v_0, y_0, z_0)\rangle_{L^p(\Omega)} \\ &= \langle F(v_0, y_0, z_0), g\rangle_{L^p(\Omega)}. \quad (3.26)\end{aligned}$$

Taking into account that $F(v_k, y_k, z_k) \to F(v_0, y_0, z_0)$ weakly in $L^q(\Omega)$ (see Theorem 3.1), the limit passage in (3.25) leads us to the relation

$$\begin{aligned}\lim_{k\to\infty} \big(\langle F(v_k, y_k, z_k), z_k\rangle_{L^p(\Omega)} &+ \langle F(v_k, y_k, z_k), BF(v_k, y_k, z_k)\rangle_{L^p(\Omega)}\big) \\ &= \langle F(v_0, y_0, z_0), g\rangle_{L^p(\Omega)}. \quad (3.27)\end{aligned}$$

Since the right-hand sides of (3.26) and (3.27) coincide, the lower semicontinuity of the functional $\langle B\nu, \nu\rangle_{L^p(\Omega)}$ with respect to the weak topology of $L^p(\Omega)$ and $(\mathfrak{A})$-property of operator $F : L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega) \to L^q(\Omega)$ imply

$$\begin{aligned}&\langle F(v_0, y_0, z_0), g\rangle_{L^p(\Omega)} \\ &\quad= \langle F(v_0, y_0, z_0), z_0\rangle_{L^p(\Omega)} + \langle F(v_0, y_0, z_0), BF(v_0, y_0, z_0)\rangle_{L^p(\Omega)} \\ &\quad= \lim_{k\to\infty} \Big[\langle F(v_k, y_k, z_k), z_k\rangle_{L^p(\Omega)} + \langle F(v_k, y_k, z_k), BF(v_k, y_k, z_k)\rangle_{L^p(\Omega)}\Big] \\ &\quad\geqslant \liminf_{k\to\infty} \Big[\langle F(v_k, y_k, z_k), z_k\rangle_{L^p(\Omega)} + \langle F(v_k, y_k, z_k), BF(v_k, y_k, z_k)\rangle_{L^p(\Omega)}\Big] \\ &\quad\geqslant \langle F(v_0, y_0, z_0), z_0\rangle_{L^p(\Omega)} + \langle F(v_0, y_0, z_0), BF(v_0, y_0, z_0)\rangle_{L^p(\Omega)}.\end{aligned}$$

Hence,

$$\lim_{k\to\infty} \langle F(v_k, y_k, z_k), z_k\rangle_{L^p(\Omega)} = \langle F(v_0, y_0, z_0), z_0\rangle_{L^p(\Omega)},$$
$$\lim_{k\to\infty} \langle F(v_k, y_k, z_k), BF(v_k, y_k, z_k)\rangle_{L^p(\Omega)} = \langle F(v_0, y_0, z_0), BF(v_0, y_0, z_0)\rangle_{L^p(\Omega)}.$$

To conclude the proof, it remains to apply the $(\mathfrak{M})$-property of operator $F : L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega) \to L^q(\Omega)$. □

Remark 3.3. It is worth to emphasize that Corollary 3.1 leads to the following important property of Hammerstein equation (3.6)–(3.7): if the operator $F : L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega) \to L^q(\Omega)$ is compact and possesses $(\mathfrak{M})$ and $(\mathfrak{A})$

properties, then the solution set $\mathcal{H}(v,y)$ of (3.6)–(3.7) is compact with respect to the strong topology in $L^p(\Omega)$ for every pair $(v,y) \in L^\infty(D) \times W_0^{1,p}(\Omega)$. Indeed, the validity of this assertion immediately follows from Corollary 3.1 if we apply it to the sequence $\{(v_k, y_k) \equiv (v,y)\}_{k\in\mathbb{N}}$ and make use of the weak compactness property of $\mathcal{H}(v,y)$.

Now we are in a position to prove the existence result for the original optimal control problem (3.3)–(3.7).

Theorem 3.2. *Assume that $U_{ad} \times V_{ad} \neq \emptyset$ and operators $B : L^q(\Omega) \to L^p(\Omega)$ and $F : L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega) \to L^q(\Omega)$ are as in Theorem 3.1. Then the optimal control problem* (3.3)–(3.7) *admits at least one solution*

$$(\mathcal{U}^{opt}, v^{opt}, y^{opt}, z^{opt}) \in \Xi_{sol} \subset L^\infty(D;\mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega),$$
$$I_\Omega(\mathcal{U}^{opt}, v^{opt}, y^{opt}, z^{opt}) = \inf_{(\mathcal{U},v,y,z)\in\Xi_{sol}} I_\Omega(\mathcal{U},v,y,z)$$

for each $f \in W^{-1,q}(D)$, $g \in L^p(D)$, and $z_d \in L^p(D)$.

Proof. Since the cost functional in (3.3) is bounded from below and, due to Theorem 2.2, the set of admissible solutions Ξ_{sol} is nonempty, it follows that there exists a sequence $\{(\mathcal{U}_k, v_k, y_k, z_k)\}_{k\in\mathbb{N}} \subset \Xi_{sol}$ such that

$$\lim_{k\to\infty} I_\Omega(\mathcal{U}_k, v_k, y_k, z_k) = \inf_{(\mathcal{U},v,y,z)\in\Xi_{sol}} I_\Omega(\mathcal{U},v,y,z).$$

As it was mentioned in Remark 3.2 the set of admissible solutions Ξ_{sol} to the problem (3.3)–(3.7) is sequentially τ-compact. Hence, there exists an admissible solution $(\mathcal{U}_0, v_0, y_0, z_0)$ such that, within a subsequence, we have $(\mathcal{U}_k, v_k, y_k, z_k) \xrightarrow{\tau} (\mathcal{U}_0, v_0, y_0, z_0)$ as $k \to \infty$. In order to show that $(\mathcal{U}_0, v_0, y_0, z_0)$ is an optimal solution of problem (3.3)–(3.6), it remains to make use of the lower semicontinuity of the cost functional with respect to the τ-convergence

$$\begin{aligned} I_\Omega(\mathcal{U}_0, v_0, y_0, z_0) &\leqslant \liminf_{m\to\infty} I_\Omega(\mathcal{U}_{k_m}, v_{k_m}, y_{k_m}, z_{k_m}) \\ &= \lim_{k\to\infty} I_\Omega(\mathcal{U}_k, v_k, y_k, z_k) = \inf_{(\mathcal{U},v,y,z)\in\Xi_{sol}} I_\Omega(\mathcal{U},v,y,z). \end{aligned}$$

The proof is complete. □

3.1. Example

In this subsection we give an example of the set $V_{ad} \subset L^\infty(D)$ and operator F for which all preconditions of Theorems 3.1,3.2 and Corollary (3.1) hold true.

Let γ, and m be given positive constants such that $\alpha|D| \leqslant m \leqslant \beta|D|$. We define the set V_{ad} as follows

$$V_{ad} = \Big\{ v \in BV(D) \cap L^\infty(D) \ \Big| \ TV(v) \leqslant \gamma,\ \|v\|_{L^1(D)} = m,\ \alpha \leqslant v(x) \leqslant \beta \text{ a.e. in } D \Big\}. \quad (3.28)$$

It is clear that V_{ad} is a nonempty convex subset of $L^1(D)$ with empty topological interior.

Proposition 3.3. If $\{v_k\}_{k\in\mathbb{N}} \subset V_{ad}$ and $v_k \to v$ strongly in $L^1(D)$, then $v_k \to v$ strongly in $L^r(D)$ for any $r \in [1, +\infty)$ and $v_k \to v$ weakly-$*$ in $L^\infty(D)$.

Proof. Since $v_k \to v$ in $L^1(D)$ and

$$\int_\Omega v_k\,dx = m, \quad TV(v_k) \leqslant \gamma, \quad \text{and} \quad \alpha \leqslant v_k \leqslant \beta \;\text{ a.e. in }\; D, \quad \forall\, k \in \mathbb{N},$$

by Proposition 2.1(i) it follows that

$$TV(v) \leqslant \gamma, \quad \int_D v\,dx = m, \quad \text{and} \quad \alpha \leqslant v \leqslant \beta \;\text{ a.e. in }\; \Omega.$$

Hence, $v \in V_{ad}$. Moreover, for any $r \in [1, +\infty)$, the estimate

$$\|v_k - v\|^r_{L^r(\Omega)} \leqslant \operatorname*{vrai\,sup}_{x\in D} |v_k(x) - v(x)\|^{r-1} \|v_k - v\|_{L^1(D)} \leqslant (\beta - \alpha)^{r-1} \|v_k - v\|_{L^1(D)}$$

implies that $v_k \to v$ strongly in $L^r(D)$.

To end the proof, it is enough to note that strong convergence $v_k \to v$ in $L^1(D)$ implies, up to a subsequence, convergence $v_k(x) \to v(x)$ almost everywhere in D. Hence, by Lebesgue Theorem, we have

$$\int_\Omega (v_k - v)\varphi\,dx \to 0, \quad \forall\, \varphi \in L^1(\Omega),$$

that is $v_k \to v$ weakly-$*$ in $L^\infty(D)$. Since this conclusion is true for any weakly-$*$ convergent subsequence of $\{v_k\}_{k\in\mathbb{N}}$, it follows that u is the weak-$*$ limit for the whole sequence $\{v_k\}_{k\in\mathbb{N}}$. □

Proposition 3.4. V_{ad} is a sequentially compact subset of $L^r(D)$ for any $r \in [1, +\infty)$, and it is a sequentially weakly-$*$ compact subset of $L^\infty(D)$.

Proof. Let $\{v_k\}_{k\in\mathbb{N}}$ be any sequence of V_{ad}. Then $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $BV(D) \cap L^\infty(D)$. As a result, the statement immediately follows from Propositions 3.3 and 2.1(iii). □

As an example of the nonlinear operator $F : L^\infty(D) \times W_0^{1,p}(\Omega) \times L^p(\Omega) \to L^q(\Omega)$ satisfying all conditions of Theorem 3.1 and Corollary 3.1, we can consider the following one

$$F(v, y, z) = |y|^{p-2}y + v(x)|z|^{p-2}z.$$

Indeed, this function is obviously radially continuous with respect to the third argument and it is also strictly monotone by z

$$\begin{aligned}
&\langle F(v, y, z_1) - F(v, y, z_2), z_1 - z_2\rangle_{L^p(\Omega)} \\
&\qquad = \int_\Omega v(x)\left(|z_1|^{p-2}z_1 - |z_2|^{p-2}z_2\right)(z_1 - z_2)\,dx \\
&\qquad\qquad \geqslant \alpha 2^{2-p}\|z_1 - z_2\|^p_{L^p(\Omega)} > 0, \;\forall\, z_1, z_2 \in L^p(\Omega),\; z_1 \neq z_2.
\end{aligned}$$

This implies that F is an operator with u.s.b.v. It is also easy to see that mapping $F : V_{ad} \times W_0^{1,p}(\Omega) \times L^p(\Omega) \to L^q(\Omega)$ is compact in a way pronounced by Theorem 3.1.

Indeed, let $y_k \to y_0$ weakly in $W_0^{1,p}(\Omega)$ and $v_k \to v_0$ weakly in $L^\infty(D)$. Then, in view of the Sobolev embedding theorem, we have $y_k \to y_0$ strongly in $L^p(\Omega)$. Combining this fact with the convergence of norms

$$\| \, |y_k|^{p-2} \, y_k\|^q_{L^q(\Omega)} = \|y_k\|^p_{L^p(\Omega)} \to \|y_0\|^p_{L^p(\Omega)} = \| \, |y_0|^{p-2} \, y_0\|^q_{L^q(\Omega)}$$

we arrive at the strong convergence $|y_k|^{p-2} y_k \to |y_0|^{p-2} y_0$ in $L^q(\Omega)$.

Also, due to Proposition 3.4, we get that within a subsequence still denoted by the same index $v_k \to v_0$ strongly in $L^1(D)$, $v_k \to v_0$ a.e. in D and $\{v_k\}_{k\in\mathbb{N}}$ is equi-integrable on $\Omega \subset D$. Further, the sequence $\{v_k(x)|z|^{p-2}z\}_{k\in\mathbb{N}}$ is bounded in $L^q(\Omega)$ and hence weakly compact, namely $v_k(x)|z|^{p-2}z \to v_0(x)|z|^{p-2}z$ weakly in $L^q(\Omega)$. Moreover, by Lebesgue Theorem we have the following convergence of norms

$$\int_\Omega v_k^q(x) \left||z|^{p-2}z\right|^q dx = \int_\Omega v_k^q(x)|z|^p \, dx \to \int_\Omega v_0^q(x)|z|^p \, dx,$$

since the sequence $\{v_k^q(x)|z|^p\}_{k\in\mathbb{N}}$ is obviously bounded in $L^1(\Omega)$, equi-integrable and converges a.e. in $\Omega \subset D$.

As a result, we have $F(v_k, y_k, z) \to F(v_0, y_0, z)$ strongly in $L^q(\Omega)$.

Now let us show that F possesses the $(\mathfrak{M})$ and $(\mathfrak{A})$ properties. Let $v_k \to v$ strongly in $L^\infty(D)$, $y_k \to y$ strongly in $W_0^{1,p}(\Omega)$ and $z_k \to z$ weakly in $L^p(\Omega)$. First, let us prove that condition (2.13) holds true. Indeed, the following chain of relations

$$\begin{aligned}
&\liminf_{k\to\infty} \langle z_k, F(v_k, y_k, z_k)\rangle_{L^p(\Omega)} \\
&\qquad\geqslant \lim_{k\to\infty} \langle |y_k|^{p-2} y_k, z_k\rangle_{L^p(\Omega)} + \liminf_{k\to\infty} \langle v_k |z_k|^{p-2} z_k, z_k\rangle_{L^p(\Omega)} \\
&\quad\geqslant \langle |y|^{p-2} y, z\rangle_{L^p(\Omega)} + \lim_{k\to\infty} \int_\Omega (v_k - v)|z_k|^p \, dx + \liminf_{k\to\infty} \int_\Omega v|z_k|^p \, dx \\
&\qquad\qquad\geqslant \langle |y|^{p-2} y, z\rangle_{L^p(\Omega)} + \int_\Omega v|z|^p \, dx = \langle F(v, y, z), z\rangle_{L^p(\Omega)}, \quad (3.29)
\end{aligned}$$

takes place in view of Lebesgue Theorem (since the sequence $\{(v_k - v)|z_k|^p\}_{k\in\mathbb{N}}$ is equi-integrable and converges to zero a.e. in Ω) and the fact that the expression

$$|||\xi|||_{L^p(\Omega)} = \left(\int_\Omega v(x)|\xi(x)|^p \, dx\right)^{1/p}$$

can be taken as an equivalent norm of element ξ in $L^p(\Omega)$. Hence the $(\mathfrak{A})$ property holds true for operator F.

Taking into account condition (2.12) let us prove the strong convergence $z_k \to z$ in $L^p(\Omega)$. It is easy to see, that changing everywhere in (3.29) $\liminf_{k\to\infty}$ into $\lim_{k\to\infty}$ and " $\geqslant$ " to $-$", we obtain the relation which implies the norm convergence $|||z_k|||_{L^p(\Omega)} \to |||z|||_{L^p(\Omega)}$. Since $z_k \to z$ weakly in $L^p(\Omega)$, we finally conclude: the sequence $\{z_k\}_{k\in\mathbb{N}}$ is strongly convergent to z in $L^p(\Omega)$.

4. Domain Perturbations for Optimal Control Problem

The aim of this section is to study the asymptotic behavior of solutions $(\mathcal{U}_\varepsilon^{opt}, v_\varepsilon^{opt}, y_\varepsilon^{opt}, z_\varepsilon^{opt})$ to the optimal control problems

$$I_{\Omega_\varepsilon}(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) = \int_{\Omega_\varepsilon} |z_\varepsilon(x) - z_d(x)|^p \, dx \longrightarrow \inf, \tag{4.1}$$

$$-\mathrm{div}\, \big(\mathcal{U}_\varepsilon(x)[(\nabla y_\varepsilon)^{p-2}]\nabla y_\varepsilon\big) + |y_\varepsilon|^{p-2} y_\varepsilon = f \text{ in } \Omega_\varepsilon, \tag{4.2}$$

$$y_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon), \quad \mathcal{U}_\varepsilon \in U_{ad}, \tag{4.3}$$

$$z_\varepsilon + BF(v_\varepsilon, y_\varepsilon, z_\varepsilon) = g \text{ in } \Omega_\varepsilon, \ z_\varepsilon \in L^p(\Omega_\varepsilon), \tag{4.4}$$

$$v_\varepsilon \in V_{ad}, \tag{4.5}$$

as $\varepsilon \to 0$ under some appropriate perturbations $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of a fixed domain $\Omega \subseteq D$. As before, we suppose that $f \in W^{-1,q}(D)$, $g \in L^p(D)$, and $z_d \in L^p(D)$ are given functions. We assume that the set of admissible controls $U_{ad} \times V_{ad}$ and, hence, the corresponding sets of admissible solutions $\Xi_\varepsilon \subset L^\infty(D; \mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(\Omega_\varepsilon) \times L^p(\Omega_\varepsilon)$ are nonempty for every $\varepsilon > 0$. We also assume that all conditions of Theorem 3.1 and Corollary 3.1 hold true for every open subset Ω of D.

The following assumption is crucial for our further analysis.

($\mathfrak{B}$) The Hammerstein equation

$$\int_D z\,\phi\,dx + \int_D BF(v,y,z)\,\phi\,dx = \int_D g\,\phi\,dx, \tag{4.6}$$

possesses property ($\mathfrak{B}$), i.e. for any triplet $(v, y, z) \in L^\infty(D) \times W_0^{1,p}(D) \times L^p(D)$ such that $z \in \mathcal{H}(v, y)$ and any sequence $\{y_k\}_{k\in\mathbb{N}} \subset W_0^{1,p}(D)$, strongly convergent in $W_0^{1,p}(D)$ to the element y, there exists a sequence $\{z_k\}_{k\in\mathbb{N}} \subset L^p(D)$ such that

$$z_k \in \mathcal{H}(v, y_k), \quad \forall\, k \in \mathbb{N} \quad \text{ and } \quad z_k \to z \text{ strongly in } L^p(D).$$

Remark 4.1. As we have already mentioned in Remark 3.3, under assumptions of Corollary 3.1, the set $\mathcal{H}(v, y)$ is non-empty and compact with respect to strong topology of $L^p(D)$ for every pair $(v, y) \in L^\infty(D) \times W_0^{1,p}(D)$. Hence, the ($\mathfrak{B}$)-property obviously holds true provided $\mathcal{H}(v, y)$ is a singleton (even if each of the sets $\mathcal{H}(v, y_k)$ contains more than one element).

Before we give the precise definition of the shape stability for the above problem and admissible perturbations for open set Ω, we remark that neither the set convergence $\Omega_\varepsilon \xrightarrow{H^c} \Omega$ in the Hausdorff complementary topology nor the topological set convergence $\Omega_\varepsilon \xrightarrow{\text{top}} \Omega$ is a sufficient condition to prove the shape stability of the control problem (3.3)–(3.7). In general, a limit quadruple for the sequence $\left\{(\mathcal{U}_\varepsilon^{opt}, v_\varepsilon^{opt}, y_\varepsilon^{opt}, z_\varepsilon^{opt})\right\}_{\varepsilon>0}$, under H^c-perturbations of Ω, can be non-admissible to the original problem (3.3)–(3.7). We refer to [6] for simple counterexamples. So, we have to impose some additional constraints on the moving domain. In view of this, we begin with the following concepts:

Definition 4.1. Let Ω and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be open subsets of D. We say that the sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ form an H^c-admissible perturbation of Ω, if:

(i) $\Omega_\varepsilon \xrightarrow{H^c} \Omega$ as $\varepsilon \to 0$;

(ii) $\Omega_\varepsilon \in \mathcal{W}_w(D)$ for every $\varepsilon > 0$, where the class $\mathcal{W}_w(D)$ is defined in (2.4).

Definition 4.2. Let Ω and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be open subsets of D. We say that sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ form a topologically admissible perturbation of Ω (shortly, t-admissible), if $\Omega_\varepsilon \xrightarrow{\text{top}} \Omega$ in the sense of Definition 2.8.

Remark 4.2. As Theorem 2.5 indicates, a subset $\Omega \subset D$ admits the existence of H^c-admissible perturbations if and only if Ω belongs to the family $\mathcal{W}_w(D)$. It turns out that the assertion:

$$\text{“ } y \in W^{1,p}(\mathbb{R}^N),\ \Omega \in \mathcal{W}_w(D), \quad \text{and} \quad \operatorname{supp} y \subset \overline{\Omega}, \quad \text{imply} \quad y \in W_0^{1,p}(\Omega) \text{ ”}$$

is not true, in general. In particular, the above statement does not take place in the case when an open domain Ω has a crack. So, $\mathcal{W}_w(D)$ is a rather general class of open subsets of D.

Remark 4.3. The remark above motivates us to say that we call $\Omega \subset D$ a p-stable domain if for any $y \in W^{1,p}(\mathbb{R}^N)$ such that $y = 0$ almost everywhere on $\operatorname{int}\Omega^c$, we get $y|_\Omega \in W_0^{1,p}(\Omega)$. Note that this property holds for all reasonably regular domains such as Lipschitz domains for instance. A more precise discussion of this property may be found in [8].

Hereinafter, we denote by $\widetilde{y}_\varepsilon$ the zero-extension of y_ε to $\mathbb{R}^N$. We begin with the following result.

Proposition 4.1. Let $\Omega \in \mathcal{W}_w(D)$ be a fixed subdomain of D, and let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be an H^c-admissible perturbation of Ω. Let $\{(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible quadruples to problems (4.1)–(4.5). Then sequence $\{(\mathcal{U}_\varepsilon, v_\varepsilon, \widetilde{y}_\varepsilon, \widetilde{z}_\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(D;\mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(D) \times L^p(D)$ and for each its τ-cluster quadruple $(\mathcal{U}^*, v^*, y^*, z^*) \in L^\infty(D;\mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(D) \times$

$L^p(D)$, we have

$$\mathcal{U}^* \in U_{ad}, \tag{4.7}$$

$$\int_D \left(\mathcal{U}^*[(\nabla y^*)^{p-2}]\nabla y^*, \nabla\widetilde{\varphi}\right)_{\mathbb{R}^N}\, dx + \int_D |y^*|^{p-2} y^* \widetilde{\varphi}\, dx$$
$$= \langle f, \widetilde{\varphi}\rangle_{W_0^{1,p}(D)}, \ \forall\, \varphi \in C_0^\infty(\Omega), \tag{4.8}$$

$$\int_D z^* \widetilde{\psi}\, dx + \langle BF(v^*, y^*, z^*), \widetilde{\psi}\rangle_{L^q(D)} = \int_D g\,\widetilde{\psi}\, dx, \quad \forall\, \psi \in C_0^\infty(\Omega), \tag{4.9}$$

$$v^* \in V_{ad}. \tag{4.10}$$

Proof. Since each of the quadruples $(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon)$ is admissible to the corresponding problem (4.1)–(4.5), the uniform boundedness of sequence $\{(\mathcal{U}_\varepsilon, v_\varepsilon, \widetilde{y}_\varepsilon, \widetilde{z}_\varepsilon)\}_{\varepsilon>0}$ with respect to the norm of $L^\infty(D; \mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(D) \times L^p(D)$ is a direct consequence of (3.2), Proposition 3.1, and Theorem 3.1. So, we may assume that there exists a quadruple $(\mathcal{U}^*, v^*, y^*, z^*)$ such that (within a subsequence still denoted by suffix ε) $(\mathcal{U}_\varepsilon, v_\varepsilon, \widetilde{y}_\varepsilon, \widetilde{z}_\varepsilon) \xrightarrow{\tau} (\mathcal{U}^*, v^*, y^*, z^*)$ in $L^\infty(D; \mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(D) \times L^p(D)$. Then, in view of Remark 3.1, we have $\mathcal{U}^* \in U_{ad}$ and $v^* \in V_{ad}$.

Let us take as test functions $\varphi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(\Omega)$. Since $\Omega_\varepsilon \xrightarrow{H^c} \Omega$, then by Theorem 2.5, the Sobolev spaces $\left\{W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ converge in the sense of Mosco to $W_0^{1,p}(\Omega)$. Hence, for the functions $\varphi, \psi \in W_0^{1,p}(\Omega)$ fixed before, there exist sequences $\left\{\varphi_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ and $\left\{\psi_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ such that $\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi}$ and $\widetilde{\psi}_\varepsilon \to \widetilde{\psi}$ strongly in $W^{1,p}(D)$ (see property (M_1)). Since $(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon)$ is an admissible triplet for the corresponding problem in Ω_ε, we can write

$$\int_{\Omega_\varepsilon} \left(\mathcal{U}_\varepsilon[(\nabla y_\varepsilon)^{p-2}]\nabla y_\varepsilon, \nabla\varphi_\varepsilon\right)_{\mathbb{R}^N}\, dx + \int_{\Omega_\varepsilon} |y_\varepsilon|^{p-2} y_\varepsilon\, \varphi_\varepsilon\, dx = \langle f, \varphi_\varepsilon\rangle_{W_0^{1,p}(\Omega_\varepsilon)},$$
$$\int_{\Omega_\varepsilon} z_\varepsilon \psi_\varepsilon\, dx + \langle BF(v_\varepsilon, y_\varepsilon, z_\varepsilon), \psi_\varepsilon\rangle_{L^q(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} g\,\psi_\varepsilon\, dx,$$

and, hence,

$$\int_D \left(\mathcal{U}_\varepsilon[(\nabla \widetilde{y}_\varepsilon)^{p-2}]\nabla \widetilde{y}_\varepsilon, \nabla\widetilde{\varphi}_\varepsilon\right)_{\mathbb{R}^N}\, dx + \int_D |\widetilde{y}_\varepsilon|^{p-2} \widetilde{y}_\varepsilon\, \widetilde{\varphi}_\varepsilon\, dx = \langle f, \widetilde{\varphi}_\varepsilon\rangle_{W_0^{1,p}(D)}, \ \forall\, \varepsilon > 0, \tag{4.11}$$

$$\int_D \widetilde{z}_\varepsilon \widetilde{\psi}_\varepsilon\, dx + \langle BF(v_\varepsilon, \widetilde{y}_\varepsilon, \widetilde{z}_\varepsilon), \widetilde{\psi}_\varepsilon\rangle_{L^q(D)} = \int_D g\,\widetilde{\psi}_\varepsilon\, dx, \ \forall\, \varepsilon > 0. \tag{4.12}$$

To prove the equalities (4.8)–(4.9), we pass to the limit in the integral identities (4.11)–(4.12) as $\varepsilon \to 0$. Using the arguments from [18, 20] and Theorem 3.1, we have

$$
\begin{aligned}
\operatorname{div} u_{i\,\varepsilon} \to \operatorname{div} u_i^* &\ \text{ strongly in } \ W^{-1,q}(D),\ \forall\, i=1,\dots,n,\\
\left\{[(\nabla\widetilde{y}_\varepsilon)^{p-2}]\nabla\widetilde{y}_\varepsilon\right\}_{\varepsilon>0} &\ \text{ is bounded in } \ L^q(D;\mathbb{R}^N),\ q=p/(p-1),\\
\left\{|\widetilde{y}_\varepsilon|^{p-2}\widetilde{y}_\varepsilon\right\}_{\varepsilon>0} &\ \text{ is bounded in } \ L^q(D),\\
\{\widetilde{z}_\varepsilon\}_{\varepsilon>0} &\ \text{ is bounded in } \ L^p(D),\\
\{F(v_\varepsilon,\widetilde{y}_\varepsilon,\widetilde{z}_\varepsilon)\}_{\varepsilon>0} &\ \text{ is bounded in } \ L^p(D),\\
v_\varepsilon \to v^* &\text{ weakly} - * \text{ in } L^\infty(D),\\
\widetilde{y}_\varepsilon \to y^* \ \text{ in } \ L^p(D),\quad \widetilde{y}_\varepsilon(x) &\to y^*(x) \text{ a.e. } x\in D,\\
|\widetilde{y}_\varepsilon|^{p-2}\widetilde{y}_\varepsilon \to |y^*|^{p-2}y^* &\text{ weakly in } L^q(D),\\
\widetilde{z}_\varepsilon \to z^* &\text{ weakly in } L^p(D),\\
\exists\,\nu\in L^q(D)\ \text{ such that } \ F(v_\varepsilon,\widetilde{y}_\varepsilon,\widetilde{z}_\varepsilon) &\to \nu \text{ weakly in } L^p(D)
\end{aligned}
$$

where $\mathcal{U}_\varepsilon = [u_{1\,\varepsilon},\dots,u_{N\,\varepsilon}]$ and $\mathcal{U}^* = [u_1^*,\dots,u_N^*]$.

As for the sequence $\left\{f_\varepsilon := f - |\widetilde{y}_\varepsilon|^{p-2}\widetilde{y}_\varepsilon\right\}_{\varepsilon>0}$, it is clear that

$$f_\varepsilon \to f_0 = f - |y^*|^{p-2}y^* \quad \text{ strongly in } \quad W^{-1,\,q}(D).$$

In view of these observations and a priori estimate (3.8), it is easy to see that the sequence $\left\{\mathcal{U}_\varepsilon[(\nabla\widetilde{y}_\varepsilon)^{p-2}]\nabla\widetilde{y}_\varepsilon\right\}_{\varepsilon>0}$ is bounded in $L^q(D;\mathbb{R}^N)$. So, up to a subsequence, we may suppose that there exists a vector-function $\xi\in L^q(D;\mathbb{R}^N)$ such that

$$\mathcal{U}_\varepsilon[(\nabla\widetilde{y}_\varepsilon)^{p-2}]\nabla\widetilde{y}_\varepsilon \to \xi \quad \text{ weakly in } \ L^q(D;\mathbb{R}^N).$$

As a result, using the strong convergence $\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi}$ in $W^{1,p}(D)$ and the strong convergence $\widetilde{\psi}_\varepsilon \to \widetilde{\psi}$ in $L^p(D)$, the limit passage in the relations (4.11)–(4.12) as $\varepsilon\to 0$ gives

$$\int_D (\xi,\nabla\widetilde{\varphi})_{\mathbb{R}^N}\,dx = \int_D \left(f - |y^*|^{p-2}y^*\right)\widetilde{\varphi}\,dx, \tag{4.13}$$

$$\int_D z^*\widetilde{\psi}\,dx + \langle B\nu,\widetilde{\psi}\rangle_{L^q(D)} = \int_D g\,\widetilde{\psi}\,dx. \tag{4.14}$$

To conclude the proof it remains to note that the validity of equalities

$$\xi = \mathcal{U}^*[(\nabla y^*)^{p-2}]\nabla y^*, \tag{4.15}$$

$$\nu = F(v^*,y^*,z^*) \tag{4.16}$$

can be established in a similar manner as in [18, 20] and Theorem 3.1. □

Our next intention is to prove that every τ-cluster quadruple

$$(\mathcal{U}^*,v^*,y^*,z^*)\in L^\infty(D;\mathbb{R}^{N\times N})\times L^\infty(D)\times W_0^{1,p}(D)\times L^p(\Omega)$$

of the sequence $\{(\mathcal{U}_\varepsilon, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ is admissible to the original optimal control problem (3.3)–(3.7). With that in mind, as follows from (4.7)–(4.10), we have to show that $y^*|_\Omega \in W_0^{1,p}(\Omega)$ and $z^* \in \mathcal{H}(v^*, y^*|_\Omega)$, i.e.,

$$\int_\Omega z^* \psi\, dx + \langle BF(v^*, y^*, z^*), \psi\rangle_{L^q(\Omega)} = \int_\Omega g\, \psi\, dx,\ \forall\, \psi \in W_0^{1,p}(\Omega).$$

To this end, we give the following result (we refer to [4] for the details).

Lemma 4.1. *Let $\Omega, \{\Omega_\varepsilon\}_{\varepsilon>0} \in \mathcal{W}_w(D)$, and let $\Omega_\varepsilon \xrightarrow{H^c} \Omega$ as $\varepsilon \to 0$. Let $\mathcal{U}_0 \in M_\alpha^\beta(D)$ be a fixed matrix. Then*

$$\widetilde{v}_{\Omega_\varepsilon, h} \to \widetilde{v}_{\Omega, h} \text{ strongly in } W_0^{1,p}(D), \quad \forall\, h \in W_0^{1,p}(D), \tag{4.17}$$

where $v_{\Omega_\varepsilon, h}$ and $v_{\Omega, h}$ are the unique weak solutions to the boundary value problems

$$\left.\begin{aligned} -\mathrm{div}\, \big(\mathcal{U}_0[(\nabla v)^{p-2}]\nabla v\big) + |v|^{p-2} v = 0 \text{ in } \Omega_\varepsilon,\\ v - h \in W_0^{1,p}(\Omega_\varepsilon) \end{aligned}\right\} \tag{4.18}$$

and

$$\left.\begin{aligned} -\mathrm{div}\, \big(\mathcal{U}_0[(\nabla v)^{p-2}]\big) + |v|^{p-2} v = 0 \text{ in } \Omega,\\ v - h \in W_0^{1,p}(\Omega), \end{aligned}\right\} \tag{4.19}$$

respectively. Here, $\widetilde{v}_{\Omega_\varepsilon, h}$ and $\widetilde{v}_{\Omega, h}$ are the extensions of $v_{\Omega_\varepsilon, h}$ and $v_{\Omega, h}$ such that they coincide with h out of Ω_ε and Ω, respectively.

Remark 4.4. In general, Lemma 4.1 is not valid if $\Omega_\varepsilon \xrightarrow{\text{top}} \Omega$.

We are now in a position to prove the following property.

Proposition 4.2. Let $\{(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be an arbitrary sequence of admissible solutions to the family of optimal control problems (4.1)–(4.5), where $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is some H^c-admissible perturbation of the set $\Omega \in \mathcal{W}_w(D)$. If for a subsequence of $\{(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ (still denoted by the same index ε) we have $(\mathcal{U}_\varepsilon, v_\varepsilon, \widetilde{y}_\varepsilon, \widetilde{z}_\varepsilon) \xrightarrow{\tau} (\mathcal{U}^*, v^*, y^*, z^*)$, then

$$y^* = \widetilde{y}_{\Omega,\mathcal{U}^*}, \quad z^*|_\Omega \in \mathcal{H}(v^*, y_{\Omega,\mathcal{U}^*}), \tag{4.20}$$

$$\int_\Omega z^* \psi\, dx + \langle BF(v^*, y_{\Omega,\mathcal{U}^*}, z^*), \psi\rangle_{L^q(\Omega)} = \int_\Omega g\, \psi\, dx,\ \forall\, \psi \in W_0^{1,p}(\Omega), \tag{4.21}$$

$$(\mathcal{U}^*, v^*, y^*|_\Omega, z^*|_\Omega) \in \Xi_{sol}, \tag{4.22}$$

where by $y_{\Omega,\mathcal{U}^*}$ we denote the weak solution of the boundary value problem (3.4)–(3.5) with $\mathcal{U} = \mathcal{U}^*$.

Proof. To begin with, we note that, by Propositions 3.1 and 4.1, we can extract a subsequence of $\{(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ (still denoted by the same index) such that

$$\mathcal{U}_\varepsilon \to \mathcal{U}^* = [u_1^*, \dots, u_N^*] \in U_{ad} \ \text{ weakly-}* \ \text{ in } \ L^\infty(D; \mathbb{R}^{N\times N}), \tag{4.23}$$

$$v_\varepsilon \to v^* \in V_{ad} \ \text{ weakly-}* \ \text{ in } \ L^\infty(D), \tag{4.24}$$

$$\widetilde{y}_\varepsilon \to y^* \ \text{ weakly in } \ W_0^{1,p}(D), \tag{4.25}$$

$$\widetilde{z}_\varepsilon \to z^* \ \text{ weakly in } \ L^p(\Omega), \tag{4.26}$$

$$y \in W_0^{1,p}(\Omega), \quad \widetilde{y} \in W_0^{1,p}(D).$$

Since (4.21)–(4.22) are direct consequence of (4.20), we divide the proof into two steps.

Step 1. We prove that $y^* = \widetilde{y}$. Following Bucur & Trebeschi [4], for every $\varepsilon > 0$, we consider the new boundary value problem

$$\left.\begin{array}{c} -\mathrm{div}\, \big(\mathcal{U}^*[(\nabla\varphi_\varepsilon)^{p-2}]\nabla\varphi_\varepsilon\big) + |\varphi_\varepsilon|^{p-2}\varphi_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon, \\ \varphi_\varepsilon = -y^* \ \text{ in } \ D\setminus\Omega_\varepsilon. \end{array}\right\} \tag{4.27}$$

Passing to the variational statement of (4.27), we have

$$\int_D \Big(\mathcal{U}^*[(\nabla\widetilde{\varphi}_\varepsilon)^{p-2}]\nabla\widetilde{\varphi}_\varepsilon, \nabla\widetilde{\psi}_\varepsilon\Big)_{\mathbb{R}^N} dx + \int_D |\widetilde{\varphi}_\varepsilon|^{p-2}\widetilde{\varphi}_\varepsilon\, \widetilde{\psi}_\varepsilon\, dx = 0, \quad \forall\, \psi \in C_0^\infty(\Omega_\varepsilon),\ \forall\, \varepsilon > 0. \tag{4.28}$$

Taking in (4.28) as the text function $\widetilde{\psi}_\varepsilon = \widetilde{\varphi}_\varepsilon + y^* - \widetilde{y}_\varepsilon$, we obtain

$$\int_D \Big(\mathcal{U}^*[(\nabla\widetilde{\varphi}_\varepsilon)^{p-2}]\nabla\widetilde{\varphi}_\varepsilon, \nabla\left(\widetilde{\varphi}_\varepsilon + y^* - \widetilde{y}_\varepsilon\right)\Big)_{\mathbb{R}^N} dx + \int_D |\widetilde{\varphi}_\varepsilon|^{p-2}\widetilde{\varphi}_\varepsilon\, \left(\widetilde{\varphi}_\varepsilon + y^* - \widetilde{y}_\varepsilon\right) dx = 0,\ \forall\, \varepsilon > 0. \tag{4.29}$$

Let $\varphi \in W^{1,p}(\Omega)$ be the weak solution to the problem

$$\left.\begin{array}{c} -\mathrm{div}\, \big(\mathcal{U}^*[(\nabla\varphi)^{p-2}]\nabla\varphi\big) + |\varphi|^{p-2}\varphi = 0 \quad \text{in} \quad \Omega, \\ \varphi = -y^* \ \text{ in } \ D\setminus\Omega. \end{array}\right\}$$

Then by Lemma 4.1, we have $\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi}$ strongly in $W_0^{1,p}(D)$. Hence,

$$\nabla\widetilde{\varphi}_\varepsilon \to \nabla\widetilde{\varphi} \ \text{ strongly in } \ L^p(D;\mathbb{R}^N),$$

$$\|[(\nabla\widetilde{\varphi}_\varepsilon)^{p-2}]\nabla\widetilde{\varphi}_\varepsilon\|^q_{L^q(D;\mathbb{R}^N)} = \|\nabla\widetilde{\varphi}_\varepsilon\|^p_{L^p(D;\mathbb{R}^N)} \to \|\nabla\widetilde{\varphi}\|^p_{L^p(D;\mathbb{R}^N)} = \|[(\nabla\widetilde{\varphi})^{p-2}]\nabla\widetilde{\varphi}\|^q_{L^q(D;\mathbb{R}^N)},$$

$$\nabla\widetilde{\varphi}_\varepsilon(x) \to \nabla\widetilde{\varphi}(x) \ \text{ a.e. in } D,$$

and

$$\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi} \ \text{ strongly in } \ L^p(D),$$
$$\|\,|\widetilde{\varphi}_\varepsilon|^{p-2}\,\widetilde{\varphi}_\varepsilon\|^q_{L^q(D)} = \|\widetilde{\varphi}_\varepsilon\|^p_{L^p(D)} \to \|\widetilde{\varphi}\|^p_{L^p(D)} = \|\,|\widetilde{\varphi}|^{p-2}\,\widetilde{\varphi}\|^q_{L^q(D)},$$
$$\widetilde{\varphi}_\varepsilon(x) \to \widetilde{\varphi}(x) \ \text{ a.e. in } D.$$

Since the norm convergence together with pointwise convergence imply the strong convergence, it follows that

$$\begin{aligned}
[(\nabla\widetilde{\varphi}_\varepsilon)^{p-2}]\nabla\widetilde{\varphi}_\varepsilon \to [(\nabla\widetilde{\varphi})^{p-2}]\nabla\widetilde{\varphi} & \ \text{ strongly in } \ L^q(D;\mathbb{R}^N),\\
|\widetilde{\varphi}_\varepsilon|^{p-2}\,\widetilde{\varphi}_\varepsilon \to |\widetilde{\varphi}|^{p-2}\,\widetilde{\varphi} & \ \text{ strongly in } \ L^q(D),\\
\nabla\left(\widetilde{\varphi}_\varepsilon + y^* - \widetilde{y}_\varepsilon\right) \to \nabla\widetilde{\varphi} & \ \text{ weakly in } \ L^p(D;\mathbb{R}^N) \ \text{(see (4.25))},\\
(\widetilde{\varphi}_\varepsilon + y^* - \widetilde{y}_\varepsilon) \to \widetilde{\varphi} & \ \text{ strongly in } \ L^p(D),
\end{aligned}$$

Hence, the integral identity (4.29) contains only the products of weakly and strongly convergent sequences. So, passing to the limit in (4.29) as ε tends to zero, we get

$$\int_D \left(\mathcal{U}^*[(\nabla\widetilde{\varphi})^{p-2}]\nabla\widetilde{\varphi}, \nabla\widetilde{\varphi}\right)_{\mathbb{R}^N}\,dx + \int_D |\widetilde{\varphi}|^p\,dx = 0.$$

Taking into account the properties of $\mathcal{U}^*$ prescribed above, we can consider the left-hand side of the above equation as a p-th power of norm in $W_0^{1,p}(\Omega)$, which is equivalent to (2.1). Hence, it implies that $\widetilde{\varphi} = 0$ a.e. in D. However, by definition $\widetilde{\varphi} = -y^*$ in $D \setminus \Omega$. So, $y^* = 0$ in $D \setminus \Omega$, and we obtain the required property $y_{\mathcal{U}^*,\Omega} = y^*|_\Omega \in W_0^{1,p}(\Omega)$.

Step 2. Our aim is to show that $z^*|_\Omega \in \mathcal{H}(v^*, y_{\mathcal{U}^*,\Omega})$. In view of (4.9), from Proposition (4.1), we get

$$\int_\Omega z^*\psi\,dx + \int_\Omega BF(v^*, y^*, z^*)\psi\,dx = \int_\Omega g\,\psi\,dx, \quad \forall\,\psi \in C_0^\infty(\Omega).$$

As was shown at the first step, $y^* = y_{\mathcal{U}^*,\Omega}$ on Ω, and, therefore, we can rewrite the above equality in the following way

$$\int_\Omega z^*\psi\,dx + \int_\Omega BF(v^*, y_{\mathcal{U}^*,\Omega}, z^*)\psi\,dx = \int_\Omega g\,\psi\,dx, \quad \forall\,\psi \in C_0^\infty(\Omega),$$

which implies the inclusion $z^*|_\Omega \in \mathcal{H}(v^*, y_{\mathcal{U}^*,\Omega})$. The proof is complete. □

The results given above suggest us to study the asymptotic behavior of the sequences of admissible quadruples $\{(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ for the case of t-admissible perturbations of the set Ω.

Proposition 4.3. Let $\Omega \subset D$ be some p-stable open domain. Assume that $\{(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ is a sequence of admissible quadruples for the family (4.1)–(4.5), where $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset D$ forms a t-admissible perturbation of Ω. Then $\{(\mathcal{U}_\varepsilon, v_\varepsilon, \widetilde{y}_\varepsilon, \widetilde{z}_\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(D;\mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(D) \times L^p(D)$ and for every τ-cluster triplet $(\mathcal{U}^*, v^*, y^*, z^*) \in L^\infty(D;\mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(D) \times L^p(\Omega)$ of this sequence, we have

(j) the quadruple $(\mathcal{U}^*, v^*, y^*, z^*)$ satisfies the relations (4.7)–(4.10);

(jj) the quadruple $(\mathcal{U}^*, v^*, y^*|_\Omega, z^*|_\Omega)$ is admissible to the problem (3.3)–(3.7), i.e., $y^* = \widetilde{y}_{\Omega,\mathcal{U}^*}$, $z^*|_\Omega \in \mathcal{H}(v^*, y_{\Omega,\mathcal{U}^*})$, where $y_{\Omega,\mathcal{U}^*}$ stands for the weak solution of the boundary value problem (3.4)–(3.5) under $\mathcal{U} = \mathcal{U}^*$.

Proof. Since $\Omega_\varepsilon \xrightarrow{\text{top}} \Omega$ in the sense of Definition 2.8, it follows that for any $\varphi, \psi \in C_0^\infty(\Omega \setminus K_0)$ we have $\operatorname{supp}\varphi \subset \Omega_\varepsilon$, $\operatorname{supp}\psi \subset \Omega_\varepsilon$ for all $\varepsilon > 0$ small enough. Moreover, since the set K_0 has zero p-capacity, it follows that $C_0^\infty(\Omega\setminus K_0)$ is dense in $W_0^{1,p}(\Omega)$. Therefore, the verification of item (j) can be done in an analogous way to the proof of Proposition 4.1 replacing therein the sequences $\left\{\varphi_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ and $\left\{\psi_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ by the still functions φ and ψ. As for the rest, we have to repeat all arguments of that proof.

To prove the assertion (jj), it is enough to show that $y^*|_\Omega \in W_0^{1,p}(\Omega)$. To do so, let B_0 be an arbitrary closed ball not intersecting $\overline{\Omega}\cup K_1$. Then from (4.2)–(4.3) it follows that $\widetilde{y}_\varepsilon = \widetilde{y}_{\Omega_\varepsilon,\mathcal{U}_\varepsilon} = 0$ almost everywhere in B_0 whenever the parameter ε is small enough. Since by (j) and Sobolev Embedding Theorem $\widetilde{y}_\varepsilon$ converges to y^* strongly in $L^p(D)$, it follows that the same is true for the limit function y^*. As the ball B_0 was chosen arbitrary, and K_1 is of Lebesgue measure zero, it follows that $\operatorname{supp} y^* \subset \Omega$. Then, by Fubini's Theorem, we have $\operatorname{supp} y^* \subset \overline{\Omega}$. Hence, using the properties of p-stable domains (see Remark 4.3), we just come to the desired conclusion: $y^*|_\Omega \in W_0^{1,p}(\Omega)$. The rest of the proof should be quite similar to the one of Proposition 4.2, where we showed, that $z^*|_\Omega \in \mathcal{H}(v^*, y^*|_\Omega)$. The proof is complete. □

Corollary 4.1. *Let $\{(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence such that $(\mathcal{U}_\varepsilon, v_\varepsilon) \equiv (\mathcal{U}^*, v^*)$ $\forall\, \varepsilon > 0$, where $(\mathcal{U}^*, v^*) \in U_{ad} \times V_{ad}$, is an admissible control. Let the sequence $\left\{y_{\Omega_\varepsilon,\mathcal{U}^*} \in W_0^{1,p}(\Omega_\varepsilon)\right\}_{\varepsilon>0}$ be the corresponding solutions of (4.2)–(4.3) and let $z_\varepsilon \in \mathcal{H}(v^*, y_{\Omega_\varepsilon,\mathcal{U}^*})$ be any solution of (4.4)–(4.5) for each $\varepsilon > 0$. Then, under assumptions of Proposition 4.2 or Proposition 4.3, we have that, within a subsequence still denoted by the same index ε, the following convergence takes place*

$$\widetilde{y}_{\Omega_\varepsilon,\mathcal{U}^*} \to \widetilde{y}_{\Omega,\mathcal{U}^*} \quad \textit{strongly in} \quad W_0^{1,p}(D),$$
$$\widetilde{z}_\varepsilon \to z^* \quad \textit{strongly in} \quad L^p(D), \quad \textit{and} \quad z^*|_\Omega \in \mathcal{H}(v^*, y_{\Omega,\mathcal{U}^*}).$$

Proof. As follows from Propositions 4.2 and 4.3, the sequence of admissible quadruples $\{(\mathcal{U}^*, v^*, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ is relatively τ-compact, and there exists a τ-limit quadruple $(\mathcal{U}^*, v^*, y^*, z^*)$ such that $y^*|_\Omega = y_{\Omega,\mathcal{U}^*}$ and $z^*|_\Omega \in \mathcal{H}(v^*, y_{\Omega,\mathcal{U}^*})$. Having set $y = y_{\Omega,\mathcal{U}^*}$, we prove the strong convergence of $\widetilde{y}_\varepsilon$ to $\widetilde{y}$ in $W_0^{1,p}(D)$. Then the strong convergence of z_ε to z^* in $L^p(D)$ will be ensured by Corollary 3.1.

To begin with, we prove the convergence of norms of $\widetilde{y}_\varepsilon$

$$\|\widetilde{y}_\varepsilon\|_{W^{1,p}(D)} \to \|\widetilde{y}\|_{W^{1,p}(D)} \quad \text{as} \quad \varepsilon \to 0. \tag{4.30}$$

As we already mentioned, since $\mathcal{U}^* \in U_{ad}$, we can consider as an equivalent norm in $W_0^{1,p}(D)$ the following one

$$\|y\|^{\mathcal{U}^*}_{W_0^{1,p}(D)} = \left(\int_D \big(\mathcal{U}^*[(\nabla y)^{p-2}]\nabla y, \nabla y\big)_{\mathbb{R}^N}\, dx + \int_D |y|^p\, dx \right)^{1/p}.$$

As a result, the space $\left\langle W_0^{1,p}(D), \|\cdot\|^{\mathcal{U}^*}_{W_0^{1,p}(D)} \right\rangle$ endowed with this norm is uniformly convex. Hence, instead of (4.30), we can establish that

$$\|\widetilde{y}_\varepsilon\|^{\mathcal{U}^*}_{W^{1,p}(D)} \to \|\widetilde{y}\|^{\mathcal{U}^*}_{W^{1,p}(D)} \quad \text{as} \quad \varepsilon \to 0. \tag{4.31}$$

Using the equations (3.4) and (4.2), we take as test functions $\widetilde{y}$ and $\widetilde{y}_\varepsilon$, respectively. Then, passing to the limit in (4.2), we get

$$\begin{aligned} &\lim_{\varepsilon\to 0} \left(\int_D \big(\mathcal{U}^*[(\nabla \widetilde{y}_\varepsilon)^{p-2}]\nabla \widetilde{y}_\varepsilon, \nabla \widetilde{y}_\varepsilon\big)_{\mathbb{R}^N}\, dx + \int_D |\widetilde{y}_\varepsilon|^p\, dx \right) \\ &\quad = \lim_{\varepsilon\to 0} \left(\|\widetilde{y}_\varepsilon\|^{\mathcal{U}^*}_{W^{1,p}(D)} \right)^p = \lim_{\varepsilon\to 0} \langle f, \widetilde{y}_\varepsilon \rangle_{W_0^{1,p}(D)} = \langle f, \widetilde{y} \rangle_{W_0^{1,p}(D)} \\ &\quad = \int_D \big(\mathcal{U}^*[(\nabla \widetilde{y})^{p-2}]\nabla \widetilde{y}, \nabla \widetilde{y}\big)_{\mathbb{R}^N}\, dx + \int_D |\widetilde{y}|^p\, dx = \left(\|\widetilde{y}\|^{\mathcal{U}^*}_{W^{1,p}(D)} \right)^p. \end{aligned}$$

Since (4.31) together with the weak convergence in $W_0^{1,p}(D)$ imply the strong convergence, we arrive at the required conclusion. □

5. Mosco-Stability of Optimal Control Problems

We begin this section with the following concept.

Definition 5.1. We say that the optimal control problem (3.3)–(3.7) in Ω is Mosco-stable in $L^\infty(D;\mathbb{R}^{N\times N})\times L^\infty(D)\times W_0^{1,p}(D)\times L^p(D)$ along the perturbation $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of Ω, if the following conditions are satisfied

(i) if $\left\{(\mathcal{U}^0_\varepsilon, v^0_\varepsilon, y^0_\varepsilon, z^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ is a sequence of optimal solutions to the perturbed problems (4.1)–(4.5), then this sequence is relatively τ-compact in $L^\infty(D;\mathbb{R}^{N\times N}) \times L^\infty(D) \times W_0^{1,p}(D) \times L^p(D)$;

(ii) each τ-cluster quadruple of $\left\{(\mathcal{U}^0_\varepsilon, v^0_\varepsilon, y^0_\varepsilon, z^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ is an optimal solution to the original problem (3.3)–(3.7).

Moreover, if

$$(\mathcal{U}^0_\varepsilon, v^0_\varepsilon, \widetilde{y}^0_\varepsilon, \widetilde{z}^0_\varepsilon) \xrightarrow{\ \tau\ } (\mathcal{U}^0, v^0, y^0, z^0), \tag{5.1}$$

then $(\mathcal{U}^0, v^0, y^0\big|_\Omega, z^0\big|_\Omega) \in \Xi_{sol}$ and

$$\inf_{(\mathcal{U}, v, y, z)\in \Xi_{sol}} I_\Omega(\mathcal{U}, v, y, z) = I_\Omega(\mathcal{U}^0, v^0, y^0\big|_\Omega, z^0\big|_\Omega) = \lim_{\varepsilon\to 0} \inf_{(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon)\in \Xi_\varepsilon} I_{\Omega_\varepsilon}(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon). \tag{5.2}$$

Our next intention is to derive the sufficient conditions for the Mosco-stability of optimal control problem (3.3)–(3.7).

Theorem 5.1. *Let Ω, $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be open subsets of D, and let*

$$\Xi_\varepsilon \subset L^\infty(D;\mathbb{R}^{N\times N}) \times L^\infty(D) \times W^{1,p}_0(\Omega_\varepsilon) \times L^p(\Omega_\varepsilon)$$

and

$$\Xi_{sol} \subset L^\infty(D;\mathbb{R}^{N\times N}) \times L^\infty(D) \times W^{1,p}_0(\Omega) \times L^p(\Omega)$$

be the sets of admissible solutions to optimal control problems (4.1)–(4.5) and (3.3)–(3.7), respectively. Assume that operator $F : L^\infty(D) \times W^{1,p}_0(D) \times L^p(D) \to L^q(D)$ satisfies the condition

$$F(v, y\cdot\chi_\Omega, z\cdot\chi_\Omega) = 0 \text{ for a.a. } x \in D \setminus \Omega,$$

and the distributions $z_d \in L^p(D)$ in the cost functional (3.3) and $g \in L^p(D)$ in (3.6) are such that

$$z_d(x) = z_d(x)\chi_\Omega(x), \quad g(x) = g(x)\chi_\Omega(x) \quad \text{for a.e. } x \in D. \tag{5.3}$$

Assume also that Hammerstein equation (4.6) possesses property $(\mathfrak{B})$ and at least one of the suppositions

1. *$\Omega \in \mathcal{W}_w(D)$ and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is an H^c-admissible perturbation of Ω;*
2. *Ω is a p-stable domain and $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is a t-admissible perturbation of Ω;*

holds true.

Then the following assertions are valid:

(MS_1) *if $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is a numerical sequence converging to 0, and $\{(\mathcal{U}_k, v_k, y_k, z_k)\}_{k\in\mathbb{N}}$ is a sequence satisfying*

$$(\mathcal{U}_k, v_k, y_k, z_k) \in \Xi_{\varepsilon_k}, \quad \forall\, k \in \mathbb{N}, \quad \text{and}$$
$$(\mathcal{U}_k, v_k, \widetilde{y}_k, \widetilde{z}_k) \xrightarrow{\tau} (\mathcal{U}, v, \psi, \xi) \text{ in}$$
$$L^\infty(D;\mathbb{R}^{N\times N}) \times L^\infty(D) \times W^{1,p}_0(D) \times L^p(D),$$

then there exist functions $y \in W^{1,p}_0(\Omega)$ and $z \in L^p(\Omega)$ such that $y = \psi|_\Omega$, $z = \xi|_\Omega$, $z \in \mathcal{H}(v, y)$, $(\mathcal{U}, v, y, z) \in \Xi_\Omega$, and

$$\liminf_{k\to\infty} I_{\Omega_{\varepsilon_k}}(\mathcal{U}_k, v_k, y_k, z_k) \geqslant I_\Omega(\mathcal{U}, v, y|_\Omega, z|_\Omega);$$

(MS_2) *for any admissible solution* $(\mathcal{U}, v, y, z) \in \Xi_{sol}$, *there exists a realizing sequence* $\{(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *such that*

$$\begin{aligned} &\mathcal{U}_\varepsilon \to \mathcal{U} \text{ strongly in } L^\infty(D;\mathbb{R}^{N\times N}),\\ &v_\varepsilon \to v \text{ strongly in } L^\infty(D),\\ &\widetilde{y}_\varepsilon \to \widetilde{y} \text{ strongly in } W_0^{1,p}(D),\\ &\widetilde{z}_\varepsilon \to \widetilde{z} \text{ strongly in } L^p(D),\\ &\limsup_{\varepsilon\to 0} I_{\Omega_\varepsilon}(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) \leqslant I_\Omega(\mathcal{U}, v, y, z). \end{aligned}$$

Proof. To begin with, we note that the first part of property (MS_1) is the direct consequence of Propositions 4.2 and 4.3. So, it remains to check the corresponding property for cost functionals. Indeed, since $z_k \to z$ weakly in $L^p(D)$, in view of lower weak semicontinuity of norm in $L^p(D)$, we have

$$\begin{aligned} \liminf_{k\to\infty} I_{\Omega_{\varepsilon_k}}(\mathcal{U}_k, v_k, y_k, z_k) &= \liminf_{k\to\infty} \int_D |\widetilde{z}_k - z_d|^p\,dx \geqslant \int_D |z - z_d|^p\,dx\\ &\geqslant \int_\Omega |z - z_d|^p\,dx = \int_\Omega \left|z|_\Omega - z_d\right|^p\,dx = I_\Omega(\mathcal{U}, v, y|_\Omega, z|_\Omega). \end{aligned}$$

Hence, the assertion (MS_1) holds true.

Further, we prove (MS_2). In view of our initial assumptions, the set of admissible quadruples Ξ_{sol} to the problem (3.3)–(3.7) is nonempty. Let $(\mathcal{U}, v, y, z) \in \Xi_{sol}$ be an admissible quadruple. Since the control $(\mathcal{U}, v)$ is admissible to problem (4.1)–(4.5) for every $\varepsilon > 0$, we construct the sequence $\{(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ as follows: $(\mathcal{U}_\varepsilon, v_\varepsilon) = (\mathcal{U}, v)$, $\forall\,\varepsilon > 0$ and $y_\varepsilon = y_{\Omega_\varepsilon,\mathcal{U}}$ is the corresponding solution of boundary value problem (4.2)–(4.3). As for the choice of elements z_ε, we make it later on.

Then, by Corollary 4.1, we have

$$\widetilde{y}_{\Omega_\varepsilon,\mathcal{U}} \to \widetilde{y}_{\Omega,\mathcal{U}} \ \text{ strongly in } \ W_0^{1,p}(D),$$

where $y_{\Omega,\mathcal{U}}$ is a unique solution for (3.4)–(3.5). Then the inclusion $(\mathcal{U}, v, y, z) \in \Xi_{sol}$ implies $y = y_{\Omega,\mathcal{U}}$.

By the initial assumptions $g(x) = g(x)\chi(x)$ and $F(v, \widetilde{y}, \widetilde{z}) = 0$ a.e. in $D \setminus \Omega$. Hence,

$$\int_D \widetilde{z}\psi\,dx + \int_D BF(v, \widetilde{y}, \widetilde{z})\psi\,dx = \int_D g\psi\,dx,\ \forall\,\psi \in C_0^\infty(D),$$

i.e. $\widetilde{z} \in \mathcal{H}(v, \widetilde{y}) \subset L^p(D)$. Then, in view of $(\mathfrak{B})$-property, for the given triplet $(v, \widetilde{y}, \widetilde{z})$ there exists a sequence $\{\widehat{z}_\varepsilon \in \mathcal{H}(v, \widetilde{y}_{\Omega_\varepsilon,\mathcal{U}})\}_{\varepsilon>0}$ such that $\widehat{z}_\varepsilon \to \widetilde{z}$ strongly in $L^p(\Omega)$. As a result, we can take $\{(\mathcal{U}_\varepsilon, v_\varepsilon, \widetilde{y}_\varepsilon, \widehat{z}_\varepsilon)\}$ as a realizing sequence. Moreover, in this case the desired property of the cost functional seems pretty obvious.

Indeed,

$$\limsup_{\varepsilon\to 0} I_{\Omega_\varepsilon}(\mathcal{U}_\varepsilon, v_\varepsilon, y_\varepsilon, z_\varepsilon) = \limsup_{\varepsilon\to 0}\int_D |\widehat{z}_\varepsilon - z_d|^p\,dx = \int_D |\widetilde{z} - z_d|^p\,dx$$
$$= \int_\Omega |z - z_d|^p\,dx = I_\Omega(\mathcal{U}, y, z).$$

The proof is complete. □

Theorem 5.2. *Under the assumptions of Theorem 5.1 the optimal control problem* (3.3)–(3.7) *is Mosco-stable in* $L^\infty(D;\mathbb{R}^{N\times N})\times L^\infty(D)\times W_0^{1,p}(D)\times L^p(D)$.

Proof. In view of a priory estimates (2.3), (3.8) and (3.13), we can immediately conclude that any sequence of optimal quadruples $\left\{(\mathcal{U}^0_\varepsilon, v^0_\varepsilon, y^0_\varepsilon, z^0_\varepsilon)\in\Xi_\varepsilon\right\}_{\varepsilon>0}$ to the perturbed problems (4.1)–(4.5) is uniformly bounded and, hence, relatively τ-compact in
$L^\infty(D;\mathbb{R}^{N\times N})\times L^\infty(D)\times W_0^{1,p}(D)\times Ł^p(D)$. So, we may suppose that there exist a subsequence $\left\{(\mathcal{U}^0_{\varepsilon_k}, v^0_{\varepsilon_k}, y^0_{\varepsilon_k}, z^0_{\varepsilon_k})\right\}_{k\in\mathbb{N}}$ and a quadruple $(\mathcal{U}^*, v^*, y^*, z^*)$ such that $(\mathcal{U}^0_{\varepsilon_k}, v^0_{\varepsilon_k}, \widetilde{y}^0_{\varepsilon_k}, \widetilde{z}^0_{\varepsilon_k}) \xrightarrow{\tau} (\mathcal{U}^*, v^*, y^*, z^*)$ as $k\to\infty$. Then, by Theorem 5.1 (see property (MS_1)), we have $(\mathcal{U}^*, v^*, y^*|_\Omega, z^*|_\Omega)\in\Xi_{sol}$ and

$$\begin{aligned}\liminf_{k\to\infty}\min_{(\mathcal{U},v,y,z)\in\Xi_{\varepsilon_k}} I_{\Omega_{\varepsilon_k}}(\mathcal{U},v,y,z) &= \liminf_{k\to\infty} I_{\Omega_{\varepsilon_k}}(\mathcal{U}^0_{\varepsilon_k}, v^0_{\varepsilon_k}, y^0_{\varepsilon_k}, z^0_{\varepsilon_k})\\ &\geqslant I_\Omega(\mathcal{U}^*, v^*, y^*|_\Omega, z^*|_\Omega)\\ &\geqslant \min_{(\mathcal{U},v,y,z)\in\Xi_{sol}} I_\Omega(\mathcal{U},v,y,z) = I_\Omega(\mathcal{U}^{opt}, v^{opt}, y^{opt}, z^{opt}).\end{aligned}\tag{5.4}$$

However, due to condition (MS_2), for the optimal quadruple $(\mathcal{U}^{opt}, v^{opt}, y^{opt}, z^{opt})\in\Xi_{sol}$ there exists a realizing sequence $\left\{(\widehat{\mathcal{U}}_\varepsilon, v_\varepsilon, \widehat{y}_\varepsilon, \widehat{z}_\varepsilon)\in\Xi_\varepsilon\right\}_{\varepsilon>0}$ such that

$$(\widehat{\mathcal{U}}_\varepsilon, v_\varepsilon, \widetilde{\widehat{y}}_\varepsilon, \widetilde{\widehat{z}}_\varepsilon) \to (\mathcal{U}^{opt}, v^{opt}, \widetilde{y}^{opt}, \widetilde{z}^{opt}),\ \text{and}$$
$$I_\Omega(\mathcal{U}^{opt}, v^{opt}, y^{opt}, z^{opt}) \geqslant \limsup_{\varepsilon\to 0} I_{\Omega_\varepsilon}(\widehat{\mathcal{U}}_\varepsilon, v_\varepsilon, \widehat{y}_\varepsilon, \widehat{z}_\varepsilon).$$

Using this fact, we have

$$\begin{aligned}\min_{(\mathcal{U},v,y,z)\in\Xi_{sol}} I_\Omega(\mathcal{U},v,y,z) &= I_\Omega(\mathcal{U}^{opt}, v^{opt}, y^{opt}, z^{opt}) \geqslant \limsup_{\varepsilon\to 0} I_{\Omega_\varepsilon}(\widehat{\mathcal{U}}_\varepsilon, v_\varepsilon, \widehat{y}_\varepsilon, \widehat{z}_\varepsilon)\\ &\geqslant \limsup_{\varepsilon\to 0}\min_{(\mathcal{U},v,y,z)\in\Xi_\varepsilon} I_{\Omega_\varepsilon}(\mathcal{U},v,y,z)\\ &\geqslant \limsup_{k\to\infty}\min_{(\mathcal{U},v,y,z)\in\Xi_{\varepsilon_k}} I_{\Omega_{\varepsilon_k}}(\mathcal{U},v,y,z)\\ &= \limsup_{k\to\infty} I_{\Omega_{\varepsilon_k}}(\mathcal{U}^0_{\varepsilon_k}, v^0_{\varepsilon_k}, y^0_{\varepsilon_k}, z^0_{\varepsilon_k}).\end{aligned}\tag{5.5}$$

From this and (5.4), we deduce

$$\liminf_{k\to\infty} I_{\Omega_{\varepsilon_k}}(\mathcal{U}^0_{\varepsilon_k}, v^0_{\varepsilon_k}, y^0_{\varepsilon_k}, z^0_{\varepsilon_k}) \geqslant \limsup_{k\to\infty} I_{\Omega_{\varepsilon_k}}(\mathcal{U}^0_{\varepsilon_k}, v^0_{\varepsilon_k}, y^0_{\varepsilon_k}, z^0_{\varepsilon_k}).$$

Thus, combining the relations (5.4) and (5.5), and rewriting them in the form of equalities, we finally obtain

$$I_\Omega(\mathcal{U}^*, v^*, y^*|_\Omega, z^*|_\Omega) = I_\Omega(\mathcal{U}^{opt}, v^{opt}, y^{opt}, z^{opt}) = \min_{(\mathcal{U}, v, y, z) \in \Xi_{sol}} I_\Omega(\mathcal{U}, v, y, z), \tag{5.6}$$

$$I_\Omega(\mathcal{U}^{opt}, v^{opt}, y^{opt}, z^{opt}) = \lim_{k\to\infty} \min_{(\mathcal{U}, v, y, z) \in \Xi_{\varepsilon_k}} I_{\Omega_{\varepsilon_k}}(\mathcal{U}, v, y, z). \tag{5.7}$$

Since equalities (5.6)–(5.7) hold true for every τ-convergent subsequence of the original sequence of optimal solutions $\left\{(\mathcal{U}^0_\varepsilon, v^0_\varepsilon, y^0_\varepsilon, z^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$, it follows that the limits in (5.6)–(5.7) coincide and, therefore, $I_\Omega(\mathcal{U}^{opt}, v^{opt}, y^{opt}, z^{opt})$ is the limit of the whole sequence of minimal values

$$\left\{ I_{\Omega_\varepsilon}(\mathcal{U}^0_\varepsilon, v^0_\varepsilon, y^0_\varepsilon, z^0_\varepsilon) = \inf_{(\mathcal{U}, v, y, z) \in \Xi_\varepsilon} I_{\Omega_\varepsilon}(\mathcal{U}, v, y, z) \right\}_{\varepsilon>0}.$$

This concludes the proof. □

Remark 5.1. It is worth to emphasize that without ($\mathfrak{B}$)-property, the original optimal control problem can lose the Mosco-stability property with respect to the given type of domain perturbations. In such case there is no guarantee that each of optimal triplets to the OCP (3.3)–(3.7) can be attained through some sequence of optimal triplets to the perturbed problems (4.1)–(4.5).

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