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ON APPROXIMATION OF STATE-CONSTRAINED OPTIMAL CONTROL PROBLEM IN COEFFICIENTS FOR p-BIHARMONIC EQUATION

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Abstract. We study a Dirichlet-Navier optimal design problem for a quasi-linear monotone p-biharmonic equation with control and state constraints. The coefficient of the p-biharmonic operator we take as a design variable in $BV(\Omega)\cap L^\infty(\Omega)$. In order to handle the inherent degeneracy of the p-Laplacian and the pointwise state constraints, we use regularization and relaxation approaches. We derive existence and uniqueness of solutions to the underlying boundary value problem and the optimal control problem. In fact, we introduce a two-parameter model for the weighted p-biharmonic operator and Henig approximation of the ordering cone. Further we discuss the asymptotic behaviour of the solutions to regularized problem on each (ε, k)-level as the parameters tend to zero and infinity, respectively.

Key words: p-biharmonic problem, optimal control, control in coefficients, approximation, existence result.

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1. Introduction

The aim of this article is to study a state constrained Dirichlet-Navier optimal control problem (OCP) for a quasi-linear monotone elliptic equation, the so-called weighted p-biharmonic problem. The coefficient of the p-biharmonic operator, the weight u, we take as a control in $BV(\Omega)\cap L^\infty(\Omega)$. Since an important matter for applications is to obtain a solution to a given boundary value problem with desired properties, it leads to the reasonable questions: can we define an appropriate coefficient of p-biharmonic operator to minimize the discrepancy between a given displacement y_d and an expected solution to such problem. More precisely, we analyse the following optimal design problem, which can be regarded as an optimal control problem, for quasi-linear partial differential equation (PDE) with mixed boundary conditions

$$\text{Minimize } \left\{ I(u,y) = \int_\Omega |y-y_d|^p\,dx + \int_\Omega |Du| \right\} \tag{1.1}$$

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subject to the quasi-linear equation

$$\Delta(u|\Delta y|^{p-2}\Delta y) = f \quad \text{in} \quad \Omega, \tag{1.2}$$

$$y = \frac{\partial y}{\partial \nu} = 0 \quad \text{on} \ \ \Gamma_D, \quad y = \Delta y = 0 \quad \text{on} \ \Gamma_S, \tag{1.3}$$

the pointwise state constraints

$$0 \leqslant \frac{\partial y(s)}{\partial \nu} \leqslant \zeta^{max}(s) \quad \text{a.e. on} \ \ \Gamma_S, \tag{1.4}$$

and the design (control) constraints

$$u \in BV(\Omega) \ \text{ and } \ 0 < \alpha \leqslant \xi_1(x) \leqslant u(x) \leqslant \xi_2(x) \text{ a.e. in } \Omega. \tag{1.5}$$

Here, Γ_D and Γ_S are the disjoint part of the boundary $\partial\Omega$ ($\partial\Omega = \Gamma_D \cup \Gamma_S$), $BV(\Omega) \cap L^\infty(\Omega)$ stands for the control space, $y_d \in L^p(\Omega)$, $\xi_1, \xi_2 \in L^\infty(\Omega)$, $f \in L^{p'}(\Omega)$, and $\zeta^{max} \in L^p(\Gamma_S)$ are given distributions. Problems of this type appear for $p-$power-like elastic isotropic flat plates of uniform thickness, where the design variable u is to be chosen such that the deflection of the plate matches a given profile. The model extends the classical weighted biharmonic equation, where the weight $u = a^3$ involves the thickness a of the plate, see e.g. [8, 21, 25, 26], or u can be regarded as a rigidity parameter. The OCP (1.1)–(1.4) can be considered as a prototype of design problems for quasilinear state equations. For an interesting exposure to this subject we can refer to the monographs [8, 16, 17].

A particular feature of OCP (1.1)–(1.4) is the restriction by the pointwise constraints (1.4) in $L^p(\Gamma_S)$-space. In fact, the ordering cone of positive elements in L^p-spaces is typically non-solid, i.e. it has an empty topological interior. Following the standard multiplier rule, which gives a necessary optimality condition for local solutions to state constrained OCPs, the constraint qualifications such as the Slater condition or the Robinson condition should be applied in this case. However, these conditions cannot be verified for cones such as

$$L^p_+(\Gamma_S) = \{v \in L^p(\Gamma_S) \mid v \geqslant 0 \quad \text{a.e. in} \ \ \Omega\}$$

due to the fact that $\operatorname{int}\left(L^p_+(\Gamma_S)\right) = \emptyset$, where $\operatorname{int}(A)$ stands for the topological interior of the set A. Therefore, it would be reasonable to propose a suitable relaxation of the pointwise state constraints in the form of some inequality conditions involving a so-called Henig approximation $\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$ of the ordering cone of positive elements $L^p_+(\Gamma_S)$. Here, B is a fixed closed base of $L^p_+(\Gamma_S)$. As it was shown in our recent publication [12], due to fact that $L^p_+(\Gamma_S) \subset \left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$ for all $\varepsilon > 0$, we can replace the cone $L^p_+(\Gamma_S)$ by its approximation $\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$. As a result, it leads to some relaxation of the inequality constraints of the considered problem, and, hence, to the approximation of the feasible set to the original OCP. Hence, the solvability of a given class of OCPs can be characterized by solving the corresponding Henig relaxed problems in the limit $\varepsilon \to 0$ (for the details, we refer to [12, 13]).

The ones more characteristic feature of the OCP (1.1)–(1.4) is related with degeneracy of quasilinear differential operator $\Delta(u|\Delta y|^{p-2}\Delta y)$ as Δy tends to zero and also if u approaches zero. Moreover, when the term $u|\Delta y|^{p-2}$ is regarded as the coefficient of the harmonic operator, we also have the case of unbounded coefficients. In spite of the fact that the Control in the coefficients of elliptic problems has a long history of its own starting with work of Murat [19, 20] and Tartar [27] (see also Casas [4], where the constrained optimal control problem in the coefficients of the leading order differential expressions was first discussed in details), analogous results for the case of weighted p-biharmonic equations of the type $\Delta(u|\Delta y|^{p-2}\Delta y)$ remained open. In this paper, in order to avoid degeneracy with respect to the control u, we assume that u is bounded away from zero. For the precise statements see the next section. We leave the case of potentially degenerating controls to a future contribution. Instead, in this article, we focus on the degeneracies related to the nonlinearity. A number of regularizations have been suggested in the literature. See [22] for a discussion for what has come to be known as ε-p-Laplace problem, such as $\Delta_{u,\varepsilon,p}y := \operatorname{div}(u(\varepsilon + |\nabla y|^2)^{\frac{p-2}{2}})\nabla y$. While the ε-p-Laplacean regularizes the degeneracy as the gradients tend to zero, the term $u|\nabla y|^{p-2}$, viewed again as a coefficient for the otherwise linear problem, may grow large. Therefore, we introduce yet another regularization that leads to a sequence of monotone and bounded approximation $\mathcal{F}_k(|\Delta y|^2)$ of $|\Delta y|^2$ (see our recent publication [6], where this approach was developed for p-Laplace problem). For fixed parameter $p \in [2,\infty)$, and control u, we arrive at a two-parameter problem governed by

$$\Delta^2_{\varepsilon,k,p}y := \Delta(u(\varepsilon + \mathcal{F}_k(|\Delta y|^2))^{\frac{p-2}{2}})\Delta y.$$

Finally, we have to deal with a two-parameter family of optimal control problems in the coefficients for monotone nonlinear differential equations and Henig relaxation of the the inequality state constraints. We consequently provide the well-posedness analysis for the underlying partial differential equations as well as for the optimal control problems. After that we pass to the limits as $k \to \infty$ and $\varepsilon \to 0$. The approximations and Henig relaxation are not only considered to be useful for the mathematical analysis, but also for the purpose of numerical simulations.

2. Preliminaries

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ ($N \geqslant 2$). We assume that the boundary $\partial\Omega$ is Lipschitzian so that the unit outward normal $\nu = \nu(x)$ is well-defined for a.e. $x \in \partial\Omega$, where a.e. means here with respect to the $(N-1)$-dimensional Hausdorff measure. We also assume that the boundary $\partial\Omega$ consists of two disjoint parts $\partial\Omega = \Gamma_D \cup \Gamma_S$, where the sets Γ_D and Γ_S have positive $(N-1)$-dimensional measures, and Γ_S is now C^2. Let p be a real number such that $2 \leqslant p < \infty$.

By $W^{2,p}(\Omega)$ we denote the Sobolev space as the subspace of $L^p(\Omega)$ of functions y having generalized derivatives $D^k y$ up to order $k = 2$ in $L^p(\Omega)$. We note that due to the interpolation theory, see [1, Theorem 4.14], $W^{2,p}(\Omega)$ is a Banach space with respect to the norm

$$\|y\|_{W^{2,p}(\Omega)} = \left(\|y\|^p_{L^p(\Omega)} + \|D^2 y\|^p_{L^p(\Omega)}\right)^{1/p} = \left(\int_\Omega \left(|y|^p + |D^2 y|^p\right)\,dx\right)^{1/p},$$

where

$$D^2 y\cdot D^2 v = \left(\sum_{i_1,i_2=1}^N \frac{\partial^2 y}{\partial x_{i_1}\partial x_{i_2}}\frac{\partial^2 v}{\partial x_{i_1}\partial x_{i_2}}\right)^{1/2}, \quad \text{and} \quad |D^2 y| = \left(D^2 y\cdot D^2 y\right)^{1/2}.$$

Since $\partial\Omega$ is Lipschitzian and Γ_S is of C^2, it follows that a function $y\in W^{2,p}(\Omega)$ admits some traces on $\partial\Omega$. In particular, if ν denotes the unit outer normal to $\partial\Omega$, then for any $y\in C^2(\overline{\Omega})$ we can define the traces

$$\gamma_0(y) = y\,|_{\partial\Omega}\,,\ \ \gamma_1(y) = \left.\frac{\partial y}{\partial \nu}\right|_{\Gamma_D} \quad \text{and} \quad \gamma_2(y) = \left.\frac{\partial^2 y}{\partial \nu^2}\right|_{\Gamma_S},$$

where $\partial y/\partial\nu$ denotes the outer normal derivative of y on Γ_D defined by $\partial y/\partial\nu = (\nabla y,\nu)$. By [15, Theorem 8.3], these linear operators can be extended continuously to the space $W^{2,p}(\Omega)$. We set

$$W^{2-1/p,p}(\partial\Omega) := \gamma_0\left[W^{2,p}(\Omega)\right], \quad W^{1-1/p,p}(\Gamma_D) := \gamma_1\left[W^{2,p}(\Omega)\right]$$

as closed subspaces of $W^{1,p}(\partial\Omega)$ and $L^p(\Gamma_D)$, respectively. Since $1-1/p = 1/p'$, where p' stands for the conjugate of p (that is $p+p' = pp'$), we have $\gamma_1\left[W^{2,p}(\Omega)\right] = W^{1/p',p}(\Gamma_D)$. Moreover, the injections

$$W^{2-1/p,p}(\partial\Omega) \hookrightarrow W^{1,p}(\partial\Omega) \quad \text{and} \quad W^{1/p',p}(\Gamma_D) \hookrightarrow L^p(\Gamma_D) \tag{2.1}$$

are compact by the Sobolev embedding theorem. We also put

$$\begin{aligned}\gamma_2\left[W^{2,p}(\Omega)\right] &= W^{-1/p,p}(\Gamma_S) := \left[W^{1/p,p'}(\Gamma_S)\right]^* \\ &= \ \text{the dual space of}\ \ W^{1/p,p'}(\Gamma_S).\end{aligned}$$

Let

$$C_0^\infty(\mathbb{R}^N;\Gamma_D) = \left\{\varphi\in C_0^\infty(\mathbb{R}^N)\ :\ \begin{array}{c}\varphi = 0\ \text{ on }\ \partial\Omega,\ \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Gamma_D,\\ \text{and}\ \ \Delta\varphi = 0 \text{ on } \partial\Omega\setminus\Gamma_D.\end{array}\right\}$$

We define the Banach space $W_0^{2,p}(\Omega;\Gamma_D)$ as the closure of $C_0^\infty(\mathbb{R}^N;\Gamma_D)$ with respect to the norm $\|y\|_{W^{2,p}(\Omega)}$. Let $W^{-2,p'}(\Omega;\Gamma_D)$ be the dual space to $W_0^{2,p}(\Omega;\Gamma_D)$. We also define the space $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|y\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega \|\nabla y\|^p\,dx\right)^{1/p}$.

Throughout this paper, we use the notation $\mathbb{W}_p(\Omega) := W_0^{2,p}(\Omega;\Gamma_D)$. Let us notice that $\mathbb{W}_p(\Omega)$ equipped with the norm

$$\|y\|_{p,\Delta} := \|\Delta y\|_{L^p(\Omega)} = \left(\int_\Omega |\Delta y|^p\,dx\right)^{1/p} = \left(\int_\Omega \left|\sum_{i=1}^N \frac{\partial^2 y}{\partial x_i^2}\right|^p\,dx\right)^{1/p} \tag{2.2}$$

is a uniformly convex Banach space [3]. Moreover, the norm $\|\cdot\|_{p,\Delta}$ is equivalent on $\mathbb{W}_p(\Omega)$ to the usual norm $\|\cdot\|_{W^{2,p}(\Omega)}$ of $W^{2,p}(\Omega)$. For reader's convenience, we give below the proof of the equivalence between the standard Sobolev space norm $\|\cdot\|_{W^{2,p}(\Omega)}$ and the norm $\|\cdot\|_{p,\Delta}$. For that, let us consider the classical Dirichlet problem for the famous Poisson's equation

$$\Delta y = f \ \text{ in } \ \Omega, \quad y = 0 \ \text{ on } \ \partial\Omega. \tag{2.3}$$

Since the Laplace operator $-\Delta$ acts from $\mathbb{W}_p(\Omega)$ in $L^p(\Omega)$, it is well-known that this problem is uniquely solvable in $\mathbb{W}_p(\Omega)$ for all $f \in L^p(\Omega)$. Hence, the inverse operator $T := (-\Delta)^{-1} : L^p(\Omega) \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is well defined and satisfies the following elliptic regularity estimate [9]

$$\|Tf\|_{W^{2,p}(\Omega)} \leqslant C_p \|f\|_{L^p(\Omega)}.$$

This allows us to conclude the following. If $f \in L^p(\Omega)$ and $y \in W_0^{1,p}(\Omega)$ are such that $\frac{\partial y}{\partial \nu} = 0$ on Γ_D, $\Delta y = 0$ on Γ_S, and y is a solution of (2.3), then $-\Delta y \in L^p(\Omega)$, $y = 0$ on the boundary $\partial\Omega$, and, therefore, $y \in \mathbb{W}_p(\Omega)$. Hence,

$$\|y\|_{W^{2,p}(\Omega)} = \|T(-\Delta y)\|_{W^{2,p}(\Omega)} \leqslant C_p \|\Delta y\|_{L^p(\Omega)} = C_p \|y\|_{p,\Delta}, \tag{2.4}$$

for a suitable positive constant C_p independent of f. On the other hand, it is easy to observe that

$$\|y\|_{p,\Delta} \leqslant \|y\|_{W^{2,p}(\Omega)}.$$

Thus, by the Closed Graph Theorem, we can conclude that $y \mapsto \|y\|_{p,\Delta} = \left(\int_\Omega |\Delta y|^p\,dx\right)^{1/p}$ is equivalent to the norm induced by $W^{2,p}(\Omega)$ (for the details we refer to [7,18]).

Remark 2.1. Observe that $J : \mathbb{W}_p(\Omega) \to (\mathbb{W}_p(\Omega))^*$ defined by

$$J(y) = \begin{cases} \|\Delta y\|_{L^p(\Omega)}^{2-p} |\Delta y|^{p-2}\Delta y, & \text{if } \ y \neq 0, \\ 0, & \text{if } \ y = 0 \end{cases}$$

is the duality mapping of $\mathbb{W}_p(\Omega)$ assicuated with the norm $\|\cdot\|_{p,\Delta}$ (see [23]).

By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ for which the norm

$$\begin{aligned} \|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \int_\Omega |Df| = \|f\|_{L^1(\Omega)} \\ + \sup\left\{ \int_\Omega f \operatorname{div}\varphi\,dx \ : \ \varphi \in C_0^1(\Omega;\mathbb{R}^N), \ |\varphi(x)| \leqslant 1 \text{ for } x \in \Omega \right\} \end{aligned}$$

is finite.

We recall that a sequence $\{f_k\}_{k=1}^{\infty}$ converges weakly-$*$ to f in $BV(\Omega)$ if and only if the two following conditions hold (see [10]): $f_k \to f$ strongly in $L^1(\Omega)$ and $Df_k \rightharpoonup Df$ weakly-* in the space of Radon measures $\mathcal{M}(\Omega$, i.e.

$$\lim_{k\to\infty} \int_\Omega \varphi\, Df_k = \int_\Omega \varphi\, Df \quad \forall\, \varphi \in C_0(\Omega).$$

It is well-known also the following compactness result for BV-spaces (Helly's selection theorem, see [2]).

Theorem 2.1. *If $\{f_k\}_{k=1}^{\infty} \subset BV(\Omega)$ and $\sup_{k\in\mathbb{N}} \|f_k\|_{BV(\Omega)} < +\infty$, then there exists a subsequence of $\{f_k\}_{k=1}^{\infty}$ strongly converging in $L^1(\Omega)$ to some $f \in BV(\Omega)$ such that $Df_k \overset{*}{\rightharpoonup} Df$ weakly-$*$ in the space of Radon measures $\mathcal{M}(\Omega)$. Moreover, if $\{f_k\}_{k=1}^{\infty} \subset BV(\Omega)$ strongly converges to some f in $L^1(\Omega)$ and satisfies $\sup_{k\in\mathbb{N}} \int_\Omega |Df_k| < +\infty$, then*

$$\begin{aligned} &(i)\ f \in BV(\Omega) \quad \text{and} \quad \int_\Omega |Df| \leqslant \liminf_{k\to\infty} \int_\Omega |Df_k|;\\ &(ii)\ f_k \overset{*}{\rightharpoonup} f \quad \text{in} \quad BV(\Omega). \end{aligned} \tag{2.5}$$

3. Setting of the Optimal Control Problem

Let ξ_1, ξ_2 be fixed elements of $L^\infty(\Omega) \cap BV(\Omega)$ satisfying the conditions

$$0 < \alpha \leqslant \xi_1(x) \leqslant \xi_2(x) \text{ a.e. in } \Omega, \tag{3.1}$$

where α is a given positive value.

Let $f \in W^{-2,p'}(\Omega;\Gamma_D)$, $y_d \in L^p(\Omega)$, and $\zeta^{max} \in L^p(\Gamma_S)$ be given distributions. The optimal control problem, we consider in this paper, is to minimize the discrepancy between y_d and the solutions of the following homogeneous Dirichlet-Navier boundary valued problem

$$\Delta_p^2(u,y) = f \quad \text{in} \quad \Omega, \tag{3.2}$$

$$y = \frac{\partial y}{\partial \nu} = 0 \quad \text{on} \ \ \Gamma_D, \quad y = \Delta y = 0 \quad \text{on } \Gamma_S, \tag{3.3}$$

$$0 \leqslant \frac{\partial y(s)}{\partial \nu} \leqslant \zeta^{max}(s) \quad \text{a.e. on} \ \ \Gamma_S \tag{3.4}$$

by choosing an appropriate weight function $u \in \mathfrak{A}_{ad}$ as control. Here, $\Delta_p^2(u,\cdot)$ is the generalized p-biharmonic operator, i.e.

$$\Delta_p^2(u,y) := \Delta(u|\Delta y|^{p-2}\Delta y), \quad \Delta y = \sum_{i=1}^{N} \frac{\partial^2 y}{\partial x_i^2}$$

and the class of admissible controls $\mathfrak{A}_{ad}$ we define as follows

$$\mathfrak{A}_{ad} = \left\{ u \in L^1(\Omega) \;\middle|\; \xi_1(x) \leqslant u(x) \leqslant \xi_2(x) \text{ a.e. in } \Omega \right\}. \tag{3.5}$$

It is clear that $\mathfrak{A}_{ad}$ is a nonempty convex subset of $L^1(\Omega)$ with an empty topological interior.

More precisely, we are concerned with the following optimal control problem

$$\begin{gathered}\text{Minimize } \left\{ I(u,y) = \int_\Omega |y - y_d|^p \, dx + \int_\Omega |Du| \right\} \\ \text{subject to the constraints (3.2)–(3.5).}\end{gathered} \tag{3.6}$$

Definition 3.1. We say that an element $y \in \mathbb{W}_p(\Omega)$ is the weak solution (in the sense of Minty) to the boundary value problem (3.2)–(3.3), if

$$\int_\Omega u\Delta\varphi \left(\Delta\varphi - \Delta y\right) \, dx \geqslant \langle f, \varphi - y\rangle\,, \;\; \forall\, \varphi \in C_0^\infty(\Omega;\Gamma_D). \tag{3.7}$$

Here, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)}$ stands for the duality pairing between $(\mathbb{W}_p(\Omega))^*$ and $\mathbb{W}_p(\Omega)$ and, in the sequel, we will omit this index when it is from the context.

The existence of a unique solution to the boundary value problem (3.2)–(3.3) follows from an abstract theorem on monotone operators; see, for instance, [14] or [24, §II.2].

Theorem 3.1. *Let V be a reflexive separable Banach space. Let V^* be the dual space, and let $A : V \to V^*$ be a bounded, semicontinuous, coercive and strictly monotone operator. Then the equation $Ay = f$ has a unique solution for each $f \in V^*$. Moreover, $Ay = f$ if and only if $\langle A\varphi, \varphi - y\rangle \geqslant \langle f, \varphi - y\rangle$ for all $\varphi \in V^*$.*

Here, the above mentioned properties of the strict monotonicity, semicontinuity, and coercivity of the operator A have respectively the following meaning:

$$\langle Ay - Av, y - v\rangle_{V^*;V} \geqslant 0, \quad \forall\, y, v \in V; \tag{3.8}$$

$$\langle Ay - Av, y - v\rangle_{V^*;V} = 0 \implies y = v; \tag{3.9}$$

$$\text{the function } \; t \mapsto \langle A(y + tv), w\rangle_{V^*;V} \; \text{ is continuous for all } \; y, v, w \in V; \tag{3.10}$$

$$\lim_{\|y\|_V \to \infty} \frac{\langle Ay, y\rangle_{V^*;V}}{\|y\|_V} = +\infty. \tag{3.11}$$

In our case, we can define the operator A as a mapping $\mathbb{W}_p(\Omega) \to (\mathbb{W}_p(\Omega))^*$ by

$$\langle A\varphi, v\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)} := \int_\Omega u|\Delta\varphi|^{p-2}\Delta\varphi\Delta v \, dx.$$

Remark 3.1. The reason of such representation comes from the following observation: having applied Green's formula twice to the operator $\Delta(u|\Delta y|^{p-2}\Delta y)$

tested by $v \in C_0^\infty(\Omega;\Gamma_D)$, where y is an element of $\mathbb{W}_p(\Omega)$, we arrive at the identity

$$\begin{aligned}\int_\Omega \Delta(u|\Delta y|^{p-2}\Delta y)v\,dx &= -\int_\Omega \left(\nabla(u|\Delta y|^{p-2}\Delta y),\nabla v\right)\,dx\\ &+\int_{\partial\Omega}\frac{\partial}{\partial\nu}(u|\Delta y|^{p-2}\Delta y)v\,d\mathcal{H}^{N-1} = \int_\Omega u|\Delta y|^{p-2}\Delta y\Delta v\,dx\\ &-\int_{\Gamma_D} u|\Delta y|^{p-2}\Delta y\frac{\partial v}{\partial\nu}\,d\mathcal{H}^{N-1} - \int_{\Gamma_S} u|\Delta y|^{p-2}\Delta y\frac{\partial v}{\partial\nu}\,d\mathcal{H}^{N-1}\\ &= \int_\Omega u|\Delta y|^{p-2}\Delta y\Delta v\,dx \quad \forall\, v\in C_0^\infty(\Omega;\Gamma_D).\end{aligned}$$

Then it is easy to show that A satisfies all assumptions of Theorem 3.1 (for the details we refer to [14, 22]). As a consequence of this theorem, we also know that $y \in \mathbb{W}_p(\Omega)$ satisfies (3.7) if and only if the relations (3.2)–(3.3) are fulfilled as follows (for the details, we refer to [22, Section 2.4.4] and [8, Section 2.4.2])

$$\left.\begin{array}{c}\Delta^2(u,y) = f \quad \text{in} \quad \left(C_0^\infty(\Omega;\Gamma_D)\right)^*,\\ \gamma_0(y) = 0 \;\text{ in }\; W^{2-1/p,p}(\partial\Omega),\\ \gamma_1(y) = 0 \;\text{ in }\; W^{1/p',p}(\Gamma_D),\\ \gamma_0(\Delta y) = 0 \;\text{ in }\; W^{-1/p,p}(\Gamma_S) := \left(W^{1/p,p'}(\Gamma_S)\right)^*,\end{array}\right\}$$

that is, the integral identity holds

$$\int_\Omega u|\Delta y|^{p-2}\Delta y\Delta\varphi\,dx = \int_\Omega f\varphi\,dx \quad \forall\varphi\in\mathbb{W}_p(\Omega). \tag{3.12}$$

In particular, taking $\varphi = y$ in (3.12), this yields the relation

$$\int_\Omega u|\Delta y|^p\,dx = \int_\Omega fy\,dx, \tag{3.13}$$

which is usually referred to as the energy equality. As a result, conditions (3.1), (3.5), Friedrich's inequality, and identity (3.13) lead us to the following a priori estimate

$$\|y\|_{p,\Delta} := \left(\int_\Omega |\Delta y|^p\,dx\right)^{1/p} \leqslant C_\Omega\left(\alpha^{-1}\|f\|_{L^{p'}(\Omega)}\right)^{p'/p} \quad \forall\, u\in\mathfrak{A}_{ad}. \tag{3.14}$$

Taking this fact into account, we adopt the following notion.

Definition 3.2. We say that (u,y) is a feasible pair to the OCP (3.6) if $u \in \mathfrak{A}_{ad} \subset L^1(\Omega)$, $y \in \mathbb{W}_p(\Omega)$, the pair (u,y) is related by the integral identity (3.12), and

$$\frac{\partial y}{\partial\nu} \in L^p_+(\Gamma_S), \quad \zeta^{max} - \frac{\partial y}{\partial\nu} \in L^p_+(\Gamma_S), \tag{3.15}$$

where $L^p_+(\Gamma_S)$ stands for the natural ordering cone of positive elements in $L^p(\Gamma_S)$, i.e.

$$L^p_+(\Gamma_S) := \left\{v\in L^p(\Gamma_S)\mid v \geqslant 0 \quad \mathcal{H}^{N-1}\text{-a.e. on }\; \Gamma_S\right\}.$$

We denote by Ξ the set of all feasible pairs for the OCP (3.6).

Remark 3.2. Before we proceed further, we need to make sure that minimization problem (3.6) is consistent, i.e. there exists at least one pair (u, y) such that (u, y) satisfying the control and state constraints (3.3)–(3.5), and (u, y) would be a physically relevant solution to the boundary value problem (3.2)–(3.3). In fact, one needs the set of feasible solutions to be nonempty. But even if we are aware that $\Xi \neq \emptyset$, this set must be sufficiently rich in some sense, otherwise the OCP (3.6) becomes trivial. From a mathematical point of view, to deal directly with the control and especially state constraints is typically very difficult [4, 11, 23]. Thus, the consistency of OCPs with control and state constraints is an open question even for the simplest situation.

In view of this remark, it is reasonably now to make use of the following Hypothesis.

(H_1) OCP (3.6) is regular in the following sense — there exists at least one pair $(u, y) \in L^1(\Omega) \times \mathbb{W}_p(\Omega)$ such that $(u, y) \in \Xi$.

Let τ be the topology on the set $\Xi \subset L^1(\Omega) \times \mathbb{W}_p(\Omega)$ which we define as the product of the norm topology of $L^1(\Omega)$ and the weak topology of $W_0^{2,p}(\Omega; \Gamma_D)$. We say that a pair $(u^0, y^0) \in L^1(\Omega) \times \mathbb{W}_p(\Omega)$ is an optimal solution to problem (3.6) if

$$(u^0, y^0) \in \Xi \quad \text{and} \quad I(u^0, y^0) = \inf_{(u,y)\in\Xi} I(u, y).$$

With this notation, the control problem (3.6) can be written as follows

$$(\mathbb{P}) \qquad \min_{(u,y)\in\Xi} I(u, y).$$

4. Existence of Optimal Solutions

In this section we focus on the solvability of optimal control problem (3.2)–(3.6). Hereinafter, we suppose that the space $L^1(\Omega) \times \mathbb{W}_p(\Omega)$ is endowed with the norm $\|(u, y)\|_{L^1(\Omega)\times\mathbb{W}_p(\Omega)} := \|u\|_{L^1(\Omega)} + \|y\|_{p,\Delta}$.

We begin with a couple of auxiliary results.

Lemma 4.1. *Let $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ be a sequence such that $(u_k, y_k) \xrightarrow{\tau} (u, y)$ in $L^1(\Omega) \times \mathbb{W}_p(\Omega)$. Then we have*

$$\lim_{k\to\infty} \int_\Omega u_k \Delta y_k \Delta\varphi \, dx = \int_\Omega u \Delta y \Delta\varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega; \Gamma_D). \tag{4.1}$$

Proof. Since $u_k \to u$ in $L^1(\Omega)$ and $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, we get that $u_k \to u$ strongly in $L^r(\Omega)$ for every $1 \leqslant r < +\infty$. In particular, we have that $u_k \to u$ in $L^{p'}(\Omega)$ and $\Delta y_k \Delta\varphi \rightharpoonup \Delta y \Delta\varphi$ in $L^p(\Omega)$. Hence, it is immediate to pass to the limit and to deduce (4.1). $\square$

As a consequence, we have the following property.

Corollary 4.1. *Let* $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ *and* $\{\zeta_k \in \mathbb{W}_p(\Omega)\}_{k\in\mathbb{N}}$ *be sequences such that* $(u_k, y_k) \xrightarrow{\tau} (u, y)$ *in* $L^1(\Omega) \times \mathbb{W}_p(\Omega)$ *and* $\zeta_k \to \zeta$ *in* $\mathbb{W}_p(\Omega)$. *Then*

$$\lim_{k\to\infty} \int_\Omega u_k \Delta y_k \Delta\zeta_k \, dx = \int_\Omega u \Delta y \Delta\zeta \, dx.$$

Our next step concerns the study of topological properties of the set of feasible solutions Ξ to problem (3.6).

The following result is crucial for our further analysis.

Theorem 4.1. *Let* $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ *be a bounded sequence in* $BV(\Omega) \times \mathbb{W}_p(\Omega)$. *Then there is a pair* $(u, y) \in L^1(\Omega) \times \mathbb{W}_p(\Omega)$ *such that, up to a subsequence,* $(u_k, y_k) \xrightarrow{\tau} (u, y)$ *and* $(u, y) \in \Xi$.

Proof. By Theorem 2.1 and reflexivity of the space $\mathbb{W}_p(\Omega)$, there exists a subsequence of $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$, still denoted by the same indices, and functions $u \in BV(\Omega)$ and $y \in \mathbb{W}_p(\Omega)$ such that

$$u_k \to u \text{ in } L^1(\Omega), \quad y_k \rightharpoonup y \text{ in } \mathbb{W}_p(\Omega), \quad \text{and, hence, } y_k \rightharpoonup y \text{ in } W_0^{1,p}(\Omega). \tag{4.2}$$

Then by Lemma 4.1, we have

$$\lim_{k\to\infty} \int_\Omega u_k \Delta\varphi \Delta y_k \, dx = \int_\Omega u \Delta\varphi \Delta y \, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega; \Gamma_D).$$

It remains to show that the limit pair (u, y) is related by inequality (3.7) and satisfies the state constraints (3.15). With that in mind we write down the Minty relation for (u_k, y_k):

$$\int_\Omega u_k \Delta\varphi \left(\Delta\varphi - \Delta y_k\right) dx \geqslant \langle f, \varphi - y_k \rangle, \quad \forall\, \varphi \in C_0^\infty(\Omega; \Gamma_D). \tag{4.3}$$

In view of (4.2) and Lemma 4.1, we have

$$\lim_{k\to\infty} \int_\Omega |\Delta\varphi|^2 u_k \, dx = \int_\Omega |\Delta\varphi|^2 u \, dx, \quad \lim_{k\to\infty} \int_\Omega u_k \Delta\varphi \Delta y_k \, dx = \int_\Omega u \Delta\varphi \Delta y \, dx.$$

Thus, passing in relation (4.3) to the limit as $k \to \infty$, we arrive at the inequality (3.7) which means that $y \in \mathbb{W}_2(\Omega)$ is a weak solution to the boundary value problem (3.2)–(3.3) in the sense of Minty. Since the injections (2.1) are compact and the cone $L^p_+(\Gamma_S)$ is closed with respect to the strong convergence in $L^p(\Gamma_S)$, it follows that $\frac{\partial y_k}{\partial\nu} \to \frac{\partial y}{\partial\nu}$ strongly in $L^p(\Gamma_S)$ and, hence,

$$\lim_{k\to\infty} \gamma_1(y_k) = \gamma_1(y) \in L^p_+(\Gamma_S) \text{ and } \gamma_1(y) \in \zeta^{max} - L^p_+(\Gamma_S).$$

This fact together with $u \in \mathfrak{A}_{ad}$ leads us to the conclusion: $(u, y) \in \Xi$, i.e. the limit pair (u, y) is feasible to optimal control problem (3.6). The proof is complete. $\square$

In conclusion of this section, we give the existence result for optimal pairs to the problem (3.6).

Theorem 4.2. *Assume that, for given distributions $f \in L^{p'}(\Omega)$, $y_d \in L^p(\Omega)$, and $\zeta^{max} \in L^p(\partial\Omega)$, the Hypothesis ($H_1$) is valid. Then optimal control problem* (3.6) *admits at least one solution $(u^{opt}, y^{opt}) \in BV(\Omega) \times \mathbb{W}_p(\Omega)$.*

Proof. Since the set of feaasible pairs Ξ is nonempty and the cost functional is bounded from below on Ξ, it follows that there exists a minimizing sequence $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ to problem (3.6). Then the inequality

$$\inf_{(u,y)\in\Xi} I(u,y) = \lim_{k\to\infty} \left[\int_\Omega |y_k(x) - y_d(x)|^p \, dx + \int_\Omega |Du_k| \right] < +\infty,$$

implies the existence of a constant $C > 0$ such that

$$\sup_{k\in\mathbb{N}} \int_\Omega |Du_k| \leqslant C.$$

Hence, in view of the definition of the class of admissible controls $\mathfrak{A}_{ad}$ and a priori estimate (3.14), the sequence $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega) \times \mathbb{W}_p(\Omega)$. Therefore, by Theorem 4.1, there exist functions $u^* \in \mathfrak{A}_{ad}$ and $y^* \in \mathbb{W}_p(\Omega)$ such that $(u^*, y^*) \in \Xi$ and, up to a subsequence, $u_k \to u^*$ strongly in $L^1(\Omega)$ and $y_k \rightharpoonup y^*$ weakly in $\mathbb{W}_p(\Omega)$. To conclude the proof, it is enough to show that the cost functional I is lower semicontinuous with respect to the τ-convergence. Since $y_k \to y^*$ strongly in $L^p(\Omega)$ by Sobolev embedding theorem, it follows that

$$\lim_{k\to\infty} \int_\Omega |y_k(x) - y_d(x)|^p \, dx = \int_\Omega |y^*(x) - y_d(x)|^p \, dx,$$
$$\liminf_{k\to\infty} \int_\Omega |Du_k| \geqslant \int_\Omega |Du^*| \text{ by (2.5).}$$

Thus,

$$I(u^*, y^*) \leqslant \liminf_{k\to\infty} I(u_k, y_k) = \inf_{(u,y)\in\Xi} I(u,y).$$

Hence, (u^*, y^*) is an optimal pair, and we arrive at the required conclusion. $\square$

5. Regularization of OCP (3.6)

As was pointed out in [22], the p-Laplacian $\Delta_p(u,y)$ provides an example of a quasi-linear elliptic operator with a so-called degenerate nonlinearity for $p > 2$. In this context we have non-differentiability of the state y with respect to the control u. As follows from Theorem 4.2, this fact is not an obstacle to prove existence of optimal controls in the coefficients, but it causes certain difficulties when deriving the optimality conditions for the considered problem. On the other hand, the ordering cone of positive elements $L^p_+(\Gamma_S)$ is non-solid, i.e. it has an empty topological interior in L^p-space. Therefore, it is reasonably to apply a suitable relaxation of the pointwise state constraints in the form of some inequality conditions involving the so-called Henig approximation $\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$ of $L^p_+(\Gamma_S)$, where B is a fixed closed base of $L^p_+(\Gamma_S)$. Since $L^p_+(\Gamma_S)\subset\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$ for all $\varepsilon>0$, it allows us to replace the cone $L^p_+(\Gamma_S)$ by its approximation $\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$. In fact, it leads to some relaxation of the inequality constraints of the considered problem, and, hence, to the approximation of the feasible set to the original OCP. As a result, we introduce the following family of approximating control problems (see, for comparison, the approach of Casas and Fernandez [5] for quasi-linear elliptic equations with a distributed control in the right hand side and the approach of Kogut and Leugering [12], where the Henig regularization of pointwise state constraints have been proposed).

$$\text{Minimize}\left\{I(u,y)=\int_\Omega|y-z_d|^p\,dx+\int_\Omega|Du|\right\}\tag{5.1}$$

subject to the constraints

$$\Delta^2_{\varepsilon,k,p}(u,y)=f\quad\text{in}\quad\Omega,\tag{5.2}$$

$$y=\frac{\partial y}{\partial\nu}=0\quad\text{on}\ \ \Gamma_D,\quad y=\Delta y=0\quad\text{on}\ \Gamma_S,\tag{5.3}$$

$$\frac{\partial y}{\partial\nu}\in\left(L^p_+(\Gamma_S)\right)_\varepsilon(B),\quad\zeta^{max}-\frac{\partial y}{\partial\nu}\in\left(L^p_+(\Gamma_S)\right)_\varepsilon(B),\tag{5.4}$$

$$u\in\mathfrak{A}_{ad}=\left\{v\in BV(\Omega)\;\middle|\;\xi_1(x)\leqslant v(x)\leqslant\xi_2(x)\text{ a.e. in }\Omega\right\}.\tag{5.5}$$

Here, $k\in\mathbb{N}$, ε is a small parameter, which varies within a strictly decreasing sequence of positive numbers converging to 0,

$$\Delta^2_{\varepsilon,k,p}(u,y)=\Delta\left(u(x)\left(\varepsilon+\mathcal{F}_k\left(|\Delta y|^2\right)\right)^{\frac{p-2}{2}}\Delta y\right),\tag{5.6}$$

$\mathcal{F}_k:\mathbb{R}_+\to\mathbb{R}_+$ is a non-decreasing $C^1(\mathbb{R}_+)$-function such that

$$\begin{gathered}\mathcal{F}_k(t)=t,\ \ \text{if}\ \ t\in\left[0,k^2\right],\quad\mathcal{F}_k(t)=k^2+1,\ \ \text{if}\ \ t>k^2+1,\quad\text{and}\\ t\leqslant\mathcal{F}_k(t)\leqslant t+\delta,\ \ \text{if}\ \ k^2\leqslant t<k^2+1\quad\text{for some}\ \ \delta\in(0,1),\end{gathered}\tag{5.7}$$

$$B:=\left\{\xi\in L^p_+(\Gamma_S)\middle|\int_{\Gamma_S}\xi\,d\mathcal{H}^{N-1}=1\right\}\tag{5.8}$$

is a closed base of ordering cone $\Lambda:=L^p_+(\Gamma_S)$,

$$\begin{aligned}\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)&:=\mathrm{cl}_{\|\cdot\|_{L^p(\Gamma_S)}}\Big(\mathrm{cone}\left(B+B_\varepsilon(0)\right)\Big)\\&:=\mathrm{cl}_{\|\cdot\|_{L^p(\Gamma_S)}}\Big(\left\{\mu z\;|\;\mu\ge 0,z\in B+B_\varepsilon(0)\right\}\Big)\end{aligned}$$

is the *Henig dilating cone*, and $\frac{1}{\varepsilon}B_\varepsilon(0):=\left\{v\in L^p(\Gamma_S)\mid\|v\|_{L^p(\Gamma_S)}\leqslant 1\right\}$ is the closed unit ball in $L^p(\Gamma_S)$ centered at the origin.

As for the function $\mathcal{F}_k : \mathbb{R}_+ \to \mathbb{R}_+$, it can e.g. be defined by

$$\mathcal{F}_k(t) = \begin{cases} t, & \text{if } 0 \leqslant t \leqslant k^2, \\ (k^2-t)^3 + (k^2-t)^2 + t, & \text{if } k^2 \leqslant t \leqslant k^2+1, \\ k^2+1, & \text{if } t \geqslant k^2+1. \end{cases}$$

A direct calculation shows that in this case $\delta = 4/27$.

It is clear that the effect of such perturbations of $\Delta_p^2(u,y)$ is its regularization around critical points where $|\Delta y(x)|$ vanishes or becomes unbounded. In particular, if $y \in W_0^{2,p}(\Omega)$ and $\Omega_k(y) := \left\{ x \in \Omega \ : \ |\Delta y(x)| > \sqrt{k^2+1} \right\}$, then the following chain of inequalities

$$\begin{aligned} |\Omega_k(y)| &:= \int_{\Omega_k(y)} 1\,dx \leqslant \frac{1}{\sqrt{k^2+1}} \int_{\Omega_k(y)} |\Delta y(x)|\,dx \\ &\leqslant \frac{1}{\sqrt{k^2+1}} |\Omega_k(y)|^{\frac{1}{p'}} \left(\int_\Omega |\Delta y|^p\,dx \right)^{\frac{1}{p}} = \frac{\|y\|_{W_0^{2,p}(\Omega)}}{\sqrt{k^2+1}} |\Omega_k(y)|^{\frac{p-1}{p}} \end{aligned}$$

shows that the Lebesgue measure of the set $\Omega_k(y)$ satisfies the estimate

$$|\Omega_k(y)| \leqslant \left(\frac{1}{\sqrt{k^2+1}} \right)^p \|y\|^p_{W_0^{2,p}(\Omega)} \leqslant \|y\|^p_{W_0^{2,p}(\Omega)} k^{-p}, \quad \forall\, y \in W_0^{2,p}(\Omega), \tag{5.9}$$

i.e. the approximation $\mathcal{F}_k(|\Delta y|^2)$ is essential on sets with small Lebesgue measure. The main goal of this section is to show that for each $\varepsilon > 0$ and $k \in \mathbb{N}$, the perturbed optimal control problem (5.1)–(5.5) is well posed and its solutions can be considered as a reasonable approximation of optimal pairs to the original problem (3.6). To begin with, we establish a few auxiliary results concerning monotonicity and growth conditions for the regularized p-harmonic operator $\Delta^2_{\varepsilon,k,p}$.

For our further analysis, we make use of the following the notation

$$\|\varphi\|_{\varepsilon,k,u} = \left(\int_\Omega \left(\varepsilon + \mathcal{F}_k(|\Delta\varphi|^2) \right)^{\frac{p-2}{2}} |\Delta\varphi|^2 u\,dx \right)^{1/p} \quad \forall \varphi \in W_0^{2,2}(\Omega).$$

Remark 5.1. For an arbitrary element $y^* \in W_0^{2,2}(\Omega)$ let us consider the level set $\Omega_k(y^*) := \left\{ x \in \Omega \ : \ |\Delta y^*(x)| > \sqrt{k^2+1} \right\}$. Then

$$\begin{aligned} |\Omega_k(y^*)| &:= \int_{\Omega_k(y^*)} 1\,dx \leqslant \frac{1}{\sqrt{k^2+1}} \int_{\Omega_k(y^*)} |\Delta y^*(x)|\,dx \\ &\leqslant \frac{1}{k} |\Omega_k(y^*)|^{\frac{1}{2}} \left(\int_{\Omega_k(y^*)} |\Delta y^*|^2\,dx \right)^{\frac{1}{2}} \\ &= \frac{1}{k} \left(\frac{1}{\varepsilon + k^2 + 1} \right)^{\frac{p-2}{4}} \left(\int_{\Omega_k(y^*)} \left(\varepsilon + \mathcal{F}_k(|\Delta y^*|^2) \right)^{\frac{p-2}{2}} |\Delta y^*|^2\,dx \right)^{\frac{1}{2}} |\Omega_k(y^*)|^{\frac{1}{2}} \\ &\leqslant \frac{1}{k^{\frac{p}{2}}} \, |\Omega_k(y^*)|^{\frac{1}{2}} \, \alpha^{-\frac{1}{2}} \|y^*\|^{\frac{p}{2}}_{\varepsilon,k,u}. \end{aligned}$$

Hence, the Lebesgue measure of the set $\Omega_k(y^*)$ satisfies the estimate

$$|\Omega_k(y^*)| \leqslant \frac{\alpha^{-1}}{k^p}\|y^*\|^p_{\varepsilon,k,u}, \quad \forall\, y^* \in W^{2,2}_0(\Omega). \tag{5.10}$$

Now, we establish the following results.

Proposition 5.1. For every $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$, the operator

$$A_{\varepsilon,k,u} := -\Delta^2_{\varepsilon,k,p}(u,\cdot) : \mathbb{W}_2(\Omega) \to (\mathbb{W}_2(\Omega))^*$$

is bounded and $\|A_{\varepsilon,k,u}\| \leqslant \left(\varepsilon + k^2 + 1\right)^{\frac{p-2}{2}} \|\xi_2\|_{L^\infty(\Omega)}$, where

$$\mathbb{W}_2(\Omega) := W^{2,2}_0(\Omega;\Gamma_D).$$

Proof. From the assumptions on $\mathcal{F}_k$ and the boundedness of u we obtain

$$\begin{aligned}
\|A_{\varepsilon,k,u}\| &= \sup_{\|y\|_{W^{2,2}_0(\Omega)}\leqslant 1} \|A_{\varepsilon,k,u}y\|_{(\mathbb{W}_2(\Omega))^*}\\
&= \sup_{\|y\|_{W^{2,2}_0(\Omega)}\leqslant 1}\ \sup_{\|v\|_{W^{2,2}_0(\Omega)}\leqslant 1} \langle A_{\varepsilon,k,u}y, v\rangle_{(\mathbb{W}_2(\Omega))^*;\mathbb{W}_2(\Omega)}\\
&= \sup_{\|y\|_{W^{2,2}_0(\Omega)}\leqslant 1}\ \sup_{\|v\|_{W^{2,2}_0(\Omega)}\leqslant 1} \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\Delta y|^2)\right)^{\frac{p-2}{2}} \Delta y \Delta v u\, dx\\
&\leqslant \left(\varepsilon + k^2 + 1\right)^{\frac{p-2}{2}} \|\xi_2\|_{L^\infty(\Omega)} \sup_{\|y\|_{W^{2,2}_0(\Omega)}\leqslant 1}\ \sup_{\|v\|_{W^{2,2}_0(\Omega)}\leqslant 1} \|y\|_{W^{2,2}_0(\Omega)}\|v\|_{W^{2,2}_0(\Omega)}\\
&= \left(\varepsilon + k^2 + 1\right)^{\frac{p-2}{2}} \|\xi_2\|_{L^\infty(\Omega)},
\end{aligned}$$

which concludes the proof. □

Proposition 5.2. For every $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$, the operator $A_{\varepsilon,k,u}$ is strictly monotone.

Proof. To begin with, we make use of the following algebraic inequality:

$$\left[\left(\varepsilon + \mathcal{F}_k(|a|^2)\right)^{\frac{p-2}{2}} a - \left(\varepsilon + \mathcal{F}_k(|b|^2)\right)^{\frac{p-2}{2}} b\right](a-b) \geqslant \varepsilon^{\frac{p-2}{2}}|a-b|^2, \quad \forall\, a,b \in \mathbb{R}. \tag{5.11}$$

In order to prove it, we note that the left hand side of (5.11) can be rewritten as follows

$$\begin{aligned}
&\left(\left(\varepsilon + \mathcal{F}_k(|a|^2)\right)^{\frac{p-2}{2}} a - \left(\varepsilon + \mathcal{F}_k(|b|^2)\right)^{\frac{p-2}{2}} b\right)(a-b)\\
&\qquad = \int_0^1 \frac{d}{ds}\left\{\left(\varepsilon + \mathcal{F}_k(|sa+(1-s)b|^2)\right)^{\frac{p-2}{2}}(sa+(1-s)b)\right\} ds (a-b)\\
&= \int_0^1 \left(\varepsilon + \mathcal{F}_k(|sa+(1-s)b|^2)\right)^{\frac{p-2}{2}} |a-b|^2\, dx + (p-2)\int_0^1 \Big\{\left(\varepsilon + \mathcal{F}_k(|sa+(1-s)b|^2)\right)^{\frac{p-4}{2}}\\
&\qquad\qquad \times \mathcal{F}_k'(|sa+(1-s)b|^2)\, |(sa+(1-s)b)\,(a-b)|^2\Big\} ds = I_1 + I_2.
\end{aligned}$$

Since $p \geqslant 2$ and $\mathcal{F}_k : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing $C^1(\mathbb{R}_+)$-function, it follows that $I_2 \geqslant 0$ for all $a, b \in \mathbb{R}^N$. It remains to observe that

$$\big(\varepsilon + \mathcal{F}_k(|sa + (1-s)b|^2)\big) \geqslant \varepsilon, \quad \forall\, a, b \in \mathbb{R}.$$

Hence, $I_1 \geqslant \varepsilon^{\frac{p-2}{2}}|a-b|^2$ and we arrive at the inequality (5.11). With this we obtain

$$\begin{aligned}
&\big\langle -\Delta_{\varepsilon,k,p}(u,y) + \Delta_{\varepsilon,k,p}(u,v), y - v\big\rangle_{(\mathbb{W}_2(\Omega))^*;\mathbb{W}_2(\Omega)}\\
&= \int_\Omega u(x)\left((\varepsilon + \mathcal{F}_k(|\Delta y|^2))^{\frac{p-2}{2}}\Delta y - (\varepsilon + \mathcal{F}_k(|\Delta v|^2))^{\frac{p-2}{2}}\Delta v\right)(\Delta y - \Delta v)\,dx\\
&\geqslant \alpha\varepsilon^{\frac{p-2}{2}}\int_\Omega |\Delta y - \Delta v|^2\,dx = \alpha\varepsilon^{\frac{p-2}{2}}\|y - v\|^2_{W_0^{2,2}(\Omega)} \geqslant 0.
\end{aligned}$$

Since the relation

$$\big\langle A_{\varepsilon,k,u}y - A_{\varepsilon,k,u}v, y - v\big\rangle_{(\mathbb{W}_2(\Omega))^*;\mathbb{W}_2(\Omega)} = 0$$

implies that $y = v$ almost everywhere in Ω, it follows that the strict monotonicity property (3.9) holds in this case. $\square$

Proposition 5.3. For every $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$, the operator $A_{\varepsilon,k,u}$ is coercive (in the sense of relation (3.11)).

Proof. In order to check this property it is enough to observe that for any $y \in \mathbb{W}_2(\Omega)$, $k \in \mathbb{N}$, $\varepsilon > 0$, and $u \in \mathfrak{A}_{ad}$, we have

$$\begin{aligned}
\big\langle A_{\varepsilon,k,u}y, y\big\rangle_{(\mathbb{W}_2(\Omega))^*;\mathbb{W}_2(\Omega)} &= \big\langle -\Delta_{\varepsilon,k,p}(u,y), y\big\rangle_{(\mathbb{W}_2(\Omega))^*;\mathbb{W}_2(\Omega)}\\
&= \int_\Omega \big(\varepsilon + \mathcal{F}_k(|\Delta y|^2)\big)^{\frac{p-2}{2}}|\Delta y|^2 u\,dx \geqslant \alpha\varepsilon^{\frac{p-2}{2}}\|y\|^2_{W_0^{2,2}(\Omega)}.
\end{aligned}$$

$\square$

We are now in a position to apply the abstract theorem on monotone operators (see Theorem 3.1) to the equation $A_{\varepsilon,k,u}y = f$ with $f \in L^{p'}(\Omega)$. Closely following the arguments of Section 3, we arrive at the following assertion.

Theorem 5.1. *For each $\varepsilon > 0$, $k \in \mathbb{N}$, $u \in \mathfrak{A}_{ad}$, and $f \in L^{p'}(\Omega)$, the boundary value problem* (5.2)–(5.3) *admits a unique weak solution $y_{\varepsilon,k} \in \mathbb{W}_2(\Omega)$, i.e.*

$$\int_\Omega u(\varepsilon + \mathcal{F}_k(|\Delta y_{\varepsilon,k}|^2))^{\frac{p-2}{2}}\Delta y_{\varepsilon,k}\Delta\varphi\,dx = \int_\Omega f\varphi\,dx, \quad \forall\,\varphi \in \mathbb{W}_2(\Omega), \tag{5.12}$$

or equivalently

$$\int_\Omega u(x)(\varepsilon+\mathcal{F}_k(|\Delta\varphi|^2))^{\frac{p-2}{2}}\Delta\varphi\,(\Delta\varphi - \Delta y_{\varepsilon,k}) \tag{5.13}$$

$$\geqslant \int_\Omega f(\varphi - y_{\varepsilon,k})\,dx, \quad \forall\,\varphi \in C_0^\infty(\Omega;\Gamma_D). \tag{5.14}$$

For every $\varepsilon > 0$ and $k \in \mathbb{N}$, we denote the set of feasible pairs to the problem (5.1)–(5.5) as follows

$$\Xi_{\varepsilon,k} = \left\{ (u,y) \;\middle|\; \begin{array}{c} u \in \mathfrak{A}_{ad},\ y \in \mathbb{W}_2(\Omega), \\ (u,y) \text{ are related by equality (5.12)}, \\ \frac{\partial y}{\partial \nu} \text{ satisfies the inclusions (5.4).} \end{array} \right\}. \tag{5.15}$$

It is worth to notice that Hypothesis (H_1) about regularity of the original OCP (3.6) can be characterized by the non-emptiness properties of the sets of feasible solutions $\Xi_{\varepsilon,k}$ for approximating control problem (5.1)–(5.5). Indeed, we have the following result (see [12, Theorem 8]).

Theorem 5.2. *Let $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\delta)$ be a monotonically decreasing sequence converging to 0 as $k \to \infty$. Then, for given distributions $f \in L^{p'}(\Omega)$, $y_d \in L^p(\Omega)$, and $\zeta^{max} \in L^p(\Gamma_S)$, the Hypothesis (H_1) implies that the approximating control problem (5.1)–(5.5) has a nonempty set of feasible solutions $\Xi_{\varepsilon,k}$ for all $\varepsilon = \varepsilon_k$, $k \in \mathbb{N}$. And vice versa, if there exists a sequence $\left\{(u^k,y^k)\right\}_{k\in\mathbb{N}}$ satisfying conditions*

$$(u^k,y^k) \in \Xi_{\varepsilon_k,k} \;\; \textit{for all} \;\; k \in \mathbb{N}, \qquad \textit{and} \;\; \sup_{k\in\mathbb{N}} I(u^k,y^k) < +\infty, \tag{5.16}$$

then the sequence $\left\{(u^k,y^k)\right\}_{k\in\mathbb{N}}$ is τ-compact and each of its τ-cluster pairs is a feasible solution to the original OCP (3.6).

Thus, in view of Theorem 5.2 and Hypothesis (H_1), we can suppose that the sets $\Xi_{\varepsilon,k}$ are always nonempty and, therefore, the approximating control problem

$$(\mathbb{P}_{\varepsilon,k}) \qquad \min_{(u,y)\in\Xi_{\varepsilon,k}} I(u,y) \tag{5.17}$$

is consistent.

Analogously to problem $(\mathbb{P})$, we can prove the following theorem

Theorem 5.3. *For every positive value $\varepsilon > 0$ and integer $k \in \mathbb{N}$, the optimal control problem $(\mathbb{P}_{\varepsilon,k})$ has at least one solution.*

The proof follows the steps of that of Theorem 4.2. Indeed, it is immediate to check that $\Xi_{\varepsilon,k}$ is not empty. Then, we can take a minimizing sequence $\{(u_i,y_i)\}_{i\in\mathbb{N}} \subset \Xi_{\varepsilon,k}$. The lower boundedness of I implies the boundedness of $\{(u_i,y_i)\}_{i\in\mathbb{N}}$ in $BV(\Omega) \times W_0^{2,2}(\Omega)$. Then, arguing as in the proof of Theorem 4.2, we deduce the existence of a subsequence, denoted in the same way, and a pair $(u^*,y^*) \in \Xi_{\varepsilon,k}$ such that $u_i \overset{*}{\rightharpoonup} u^*$ in $BV(\Omega)$ and $y_i \rightharpoonup y^*$ in $W_0^{2,2}(\Omega)$. Hence, $I(u^*,y^*) \leqslant \liminf_{i\to\infty} I(u_i,y_i)$. Since $\frac{\partial y}{\partial \nu} \in W^{1/2,2}(\Gamma_S)$ for any $y \in W_0^{2,2}(\Omega;\Gamma_D)$, the injection $W^{1/2,2}(\Gamma_S) \hookrightarrow L^2(\Gamma_S)$ is compact, and the Henig dilating cone $\left(L^p_+(\Gamma_S)\right)_\varepsilon(B)$ is closed with respect to the strong convergence in $L^2(\Gamma_S)$, it follows that $\frac{\partial y_k}{\partial \nu} \to \frac{\partial y^*}{\partial \nu}$ strongly in $L^2(\Gamma_S)$ and, hence,

$$\lim_{k\to\infty} \frac{\partial y_k}{\partial \nu} = \frac{\partial y^*}{\partial \nu} \in L^p_+(\Gamma_S) \;\; \text{and} \;\; \frac{\partial y^*}{\partial \nu} \in \zeta^{max} - \left(L^p_+(\Gamma_S)\right)_\varepsilon(B).$$

This fact together with $u \in \mathfrak{A}_{ad}$ leads us to the conclusion: $(u, y) \in \Xi_{\varepsilon,k}$, i.e. the limit pair (u^*, y^*) is optimal to the problem $(\mathbb{P}_{\varepsilon,k})$.

For our further analysis, we need to obtain some appropriate a priory estimates for the weak solutions to problem (5.2)–(5.3). With that in mind, we make use of the following auxiliary results.

Proposition 5.4. Let $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$ be given. Then, for arbitrary $g \in L^2(\Omega)$ and $y \in W_0^{2,2}(\Omega)$, we have

$$\left| \int_\Omega gy\,dx \right| \leqslant C_\Omega \|g\|_{L^2(\Omega)} \left[\alpha^{-\frac{1}{p}} |\Omega|^{\frac{p-2}{2p}} \|y\|_{\varepsilon,k,u} + \alpha^{-\frac{1}{2}} \|y\|^{\frac{p}{2}}_{\varepsilon,k,u} \right]. \tag{5.18}$$

Proof. Let us fix an arbitrary element y of $W_0^{2,2}(\Omega)$. We associate with this element the set $\Omega^k(y)$, where $\Omega^k(y) := \{x \in \Omega \ : \ |\Delta y(x)| > k\}$. Then, by Friedrich's inequality,

$$\begin{aligned} \int_\Omega gy\,dx &\leqslant \|g\|_{L^2(\Omega)} \|y\|_{L^2(\Omega)} \\ &\leqslant C_\Omega \|g\|_{L^2(\Omega)} \left(\|\Delta y\|_{L^2(\Omega \setminus \Omega^k(y))} + \|\Delta y\|_{L^2(\Omega^k(y))} \right). \end{aligned} \tag{5.19}$$

Using the fact that

$$\begin{aligned} \|\Delta y\|_{L^2(\Omega\setminus\Omega^k(y))} &\leqslant |\Omega|^{\frac{p-2}{2p}} \|\Delta y\|_{L^p(\Omega\setminus\Omega^k(y))} \\ &\leqslant |\Omega|^{\frac{p-2}{2p}} \left(\int_{\Omega\setminus\Omega^k(y)} (\varepsilon + |\Delta y|^2)^{\frac{p-2}{2}} |\Delta y|^2\,dx \right)^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{gathered} \mathcal{F}_k(|\Delta y|^2) = |\Delta y|^2 \quad \text{a.e. in } \Omega \setminus \Omega^k(y), \text{ and} \\ k^2 \leqslant \mathcal{F}_k(|\Delta y|^2) \leqslant k^2 + 1 \quad \text{a.e. in } \Omega^k(y), \quad \forall\, k \in \mathbb{N}, \end{gathered}$$

we obtain

$$\begin{aligned} \|\Delta y\|_{L^2(\Omega\setminus\Omega^k(y))} &\leqslant |\Omega|^{\frac{p-2}{2p}} \left(\int_{\Omega\setminus\Omega^k(y)} (\varepsilon + \mathcal{F}_k(|\Delta y|^2))^{\frac{p-2}{2}} |\Delta y|^2\,dx \right)^{\frac{1}{p}} \\ &\leqslant |\Omega|^{\frac{p-2}{2p}} \alpha^{-\frac{1}{p}} \|y\|_{\varepsilon,k,u}, \end{aligned} \tag{5.20}$$

$$\|\Delta y\|_{L^2(\Omega^k(y))} \leqslant \left(\int_{\Omega^k(y)} (\varepsilon + \mathcal{F}_k(|\Delta y|^2))^{\frac{p-2}{2}} |\Delta y|^2\,dx \right)^{\frac{1}{2}} \leqslant \alpha^{-\frac{1}{2}} \|y\|^{\frac{p}{2}}_{\varepsilon,k,u}. \tag{5.21}$$

As a result, inequality (5.18) immediately follows from (5.19)–(5.21). The proof is complete. □

Definition 5.1. Let $\{u_{\varepsilon,k}\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \subset \mathfrak{A}_{ad}$ be an arbitrary sequence of admissible controls. We say that a two-parametric sequence $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \subset W_0^{2,2}(\Omega)$ is bounded with respect to the $\|\cdot\|_{\varepsilon,k,u_{\varepsilon,k}}$-quasi-seminorm if $\sup_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \|y_{\varepsilon,k}\|_{\varepsilon,k,u_{\varepsilon,k}} < +\infty$.

To conclude this section, let us show that for every $u \in \mathfrak{A}_{ad}$ and $f \in L^{p'}(\Omega)$, the sequence $\{y_{\varepsilon,k} = y_{\varepsilon,k}(u,f)\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ of weak solutions to the boundary value problem (5.2)–(5.3) is bounded with respect to the $\|\cdot\|_{\varepsilon,k,u}$-quasi-seminorm in the sense of Definition 5.1.

Indeed, the integral identity (5.12) together with estimate (5.18) (for $g = f$) immediately lead us to the relation

$$\begin{aligned}\|y_{\varepsilon,k}\|^p_{\varepsilon,k,u} &:= \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\Delta y_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\Delta y_{\varepsilon,k}|^2 u\,dx \\ &\leqslant \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\Delta y_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\Delta y_{\varepsilon,k}|^2 u\,dx = \int_\Omega f y_{\varepsilon,k}\,dx \\ &\leqslant C_\Omega \|f\|_{L^2(\Omega)} \left[\alpha^{-\frac{1}{p}} |\Omega|^{\frac{p-2}{2p}} \|y_{\varepsilon,k}\|_{\varepsilon,k,u} + \alpha^{-\frac{1}{2}} \|y_{\varepsilon,k}\|^{\frac{p}{2}}_{\varepsilon,k,u}\right]. \end{aligned} \tag{5.22}$$

As a result, it follows from (5.22) that

$$\|y_{\varepsilon,k}\|_{\varepsilon,k,u} \leqslant \max\left\{C_f^{\frac{2}{p}}, C_f^{\frac{1}{p-1}}\right\}, \quad \forall\,\varepsilon>0,\ \forall\,k\in\mathbb{N},\ \forall\,u\in\mathfrak{A}_{ad}, \tag{5.23}$$

where $C_f := C\|f\|_{L^2(\Omega)} = C_\Omega\left(\alpha^{-\frac{1}{p}}|\Omega|^{\frac{p-2}{2p}} + \alpha^{-\frac{1}{2}}\right)\|f\|_{L^2(\Omega)}$.

6. Asymptotic Analysis of the Approximating OCP $(\mathbb{P}_{\varepsilon,k})$

Our main intention in this section is to show that optimal solutions to the original OCP $(\mathbb{P})$ can be attained (in some sense) by optimal solutions to the approximated problems $(\mathbb{P}_{\varepsilon,k})$. With that in mind, we make use of the concept of variational convergence of constrained minimization problems (see [11]) and study the asymptotic behaviour of a family of OCPs $(\mathbb{P}_{\varepsilon,k})$ as $\varepsilon \to 0$ and $k \to \infty$. We also utilize the fact that the sequence of cones $\left\{\left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B)\right\}_{k\in\mathbb{N}}$ converges to $L^p_+(\Gamma_S)$ in Kuratowski sense with respect to the norm topology of $L^p(\Gamma_S)$ as ε_k tends monotonically to zero (see Proposition 7 in [12]), that is

$$K\text{-}\liminf_{k\to\infty} \left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B) = L^p_+(\Gamma_S) = K\text{-}\limsup_{k\to\infty} \left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B), \tag{6.1}$$

where

$$
\begin{aligned}
K-\liminf_{k\to\infty} & \left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B) \\
&:= \big\{z\in L^p(\Gamma_S)\,\big|\,\text{for all neighborhoods } N \text{ of } z \text{ there is a} \\
&\qquad k_0\in\mathbb{N} \text{ such that } N\cap\left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B)\neq\emptyset\ \forall\, k\geq k_0\big\},\\
K-\limsup_{k\to\infty} & \left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B) \\
&:= \big\{z\in L^p(\Gamma_S)\,\big|\,\text{for all neighborhoods } N \text{ of } z \text{ and every } k_0\in\mathbb{N} \\
&\qquad \text{there is a } k\geq k_0 \text{ such that } N\cap\left(L^p_+(\Gamma_S)\right)_{\varepsilon_k}(B)\neq\emptyset\big\}.
\end{aligned}
$$

We begin with some auxiliary results concerning the weak compactness in $W^{2,2}_0(\Omega)$ of $\|\cdot\|_{\varepsilon,k,u}$-bounded sequences.

Lemma 6.1. *Let $\{u_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}\subset\mathfrak{A}_{ad}$ be an arbitrary sequence of admissible controls with associated states $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}\subset W^{2,2}_0(\Omega;\Gamma_D)$, $y_{\varepsilon,k}=y_{\varepsilon,k}(u_{\varepsilon,k})$. Then the sequence $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}$ is bounded in $W^{2,2}_0(\Omega)$. Moreover, each cluster point y of the sequence $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}$ with respect to the weak convergence in $W^{2,2}_0(\Omega)$, satisfies: $y\in W^{2,p}_0(\Omega;\Gamma_D)$.*

Proof. The boundedness in $W^{2,2}_0(\Omega)$ immediately follows from (5.23) and the estimates

$$
\begin{aligned}
\|y_{\varepsilon,k}\|_{W^{2,2}_0(\Omega)} &\leqslant \|\Delta y_{\varepsilon,k}\|_{L^2(\Omega\setminus\Omega^k(y_{\varepsilon,k}))}+\|\Delta y_{\varepsilon,k}\|_{L^2(\Omega^k(y_{\varepsilon,k}))}\\
&\overset{\text{by (5.20)–(5.21)}}{\leqslant} C_\Omega\left[\alpha^{-\frac{1}{p}}|\Omega|^{\frac{p-2}{2p}}\|y_{\varepsilon,k}\|_{\varepsilon,k,u}+\alpha^{-\frac{1}{2}}\|y_{\varepsilon,k}\|^{\frac{p}{2}}_{\varepsilon,k,u}\right],
\end{aligned}
$$

where $u\in\mathfrak{A}_{ad}$ is an admissible control and $\Omega^k(y_{\varepsilon,k}):=\{x\in\Omega\ :\ |\Delta y_{\varepsilon,k}(x)|>k\}$ for each $k\in\mathbb{N}$.

To establish the second part of the lemma, let us take a subsequence $\{y_{\varepsilon_i,k_i}\}_{i\in\mathbb{N}}$ of $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}$ (here, $\varepsilon_i\to 0$ and $k_i\to\infty$ as $i\to\infty$) and a function $y\in W^{2,2}_0(\Omega;\Gamma_D)$ such that $y_{\varepsilon_i,k_i}\rightharpoonup y$ in $W^{2,2}_0(\Omega)$ as $i\to\infty$. Further, we fix an index $i\in\mathbb{N}$ and associate it with the following set

$$
B_i:=\bigcup_{j=i}^{\infty}\Omega_{k_j}(y_{\varepsilon_j,k_j}),\ \text{where}\ \ \Omega_{k_j}(y_{\varepsilon_j,k_j}):=\left\{x\in\Omega\ :\ |\Delta y_{\varepsilon_j,k_j}(x)|>\sqrt{k_j^2+1}\right\}. \tag{6.2}
$$

Due to estimates (5.10) and (5.23), we see that

$$
\begin{aligned}
|B_i| &\leqslant \alpha^{-1}\sum_{j=i}^{\infty}\frac{1}{k_j^p}\|y_{\varepsilon_j,k_j}\|^p_{\varepsilon_j,k_j,u_{\varepsilon_j,k_j}}\leqslant\alpha^{-1}\sup_{j\in\mathbb{N}}\|y_{\varepsilon_j,k_j}\|^p_{\varepsilon_j,k_j,u_{\varepsilon_j,k_j}}\sum_{j=i}^{\infty}\frac{1}{k_j^p}\\
&\leqslant\alpha^{-1}\max\left\{C_f^2,C_f^{\frac{p}{p-1}}\right\}\sum_{j=i}^{\infty}\frac{1}{k_j^p}<+\infty,
\end{aligned}
$$

and, therefore,

$$\lim_{i\to\infty} |B_i| = \mathcal{L}^N(\limsup_{i\to\infty} B_i) = 0. \tag{6.3}$$

Using again (5.23), we get

$$\begin{aligned}\int_{\Omega\setminus B_i} |\Delta y_{\varepsilon_j,k_j}|^p\,dx &\leqslant \int_{\Omega\setminus B_i} \left(\varepsilon_j + |\Delta y_{\varepsilon_j,k_j}|^2\right)^{\frac{p-2}{2}} |\Delta y_{\varepsilon_j,k_j}|^2\,dx \\ &\leqslant \alpha^{-1}\int_{\Omega\setminus B_i} \left(\varepsilon_j + \mathcal{F}_{k_j}(|\Delta y_{\varepsilon_j,k_j}|^2)\right)^{\frac{p-2}{2}} |\Delta y_{\varepsilon_j,k_j}|^2 u_{\varepsilon_j,k_j}\,dx \\ &\leqslant \alpha^{-1}\max\left\{C_f^2, C_f^{\frac{p}{p-1}}\right\}, \quad \forall j\geqslant i, \end{aligned} \tag{6.4}$$

hence $\{\Delta y_{\varepsilon_j,k_j}\}$ is bounded in $L^p(\Omega\setminus B_i)^N$. Since, $\Delta y_{\varepsilon_j,k_j} \rightharpoonup \Delta y$ in $L^2(\Omega)$, we infer $\chi_{\Omega\setminus B_j}\Delta y_{\varepsilon_j,k_j} \rightharpoonup \Delta y$ in $L^p(\Omega)$, where $\chi_{\Omega\setminus B_j}$ is the characteristic function of the set $\Omega\setminus B_j$. Hence, we obtain

$$\begin{aligned}\int_\Omega |\Delta y|^p\,dx &\overset{\text{by (6.3)}}{=} \lim_{i\to\infty}\int_{\Omega\setminus B_i} |\Delta y|^p\,dx \leqslant \lim_{i\to\infty}\liminf_{\substack{j\to\infty\\ j\geqslant i}} \int_{\Omega\setminus B_i} |\Delta y_{\varepsilon_j,k_j}|^p\,dx \\ &\overset{\text{by (6.4)}}{\leqslant} \alpha^{-1}\max\left\{C_f^2, C_f^{\frac{p}{p-1}}\right\}.\end{aligned}$$

Since $y\in W_0^{2,2}(\Omega;\Gamma_D)$, it follows from the last estimate that $y\in W_0^{2,p}(\Omega;\Gamma_D)$ and this concludes the proof. □

Lemma 6.2. *Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$, $\{k_i\}_{i\in\mathbb{N}}$ and $\{u_i\}_{i\in\mathbb{N}}\subset \mathfrak{A}_{ad}$ be sequences such that*

$$\varepsilon_i\to 0, \quad k_i\to\infty, \quad u_i\to u \text{ strongly in } L^1(\Omega).$$

Let $y_i = y_{\varepsilon_i,k_i}(u_i)$ and $y = y(u)$ be the solutions of (5.3)-(5.5) and (3.2)-(3.3), respectively. Then

$$y_i \rightharpoonup y \quad \text{in} \quad W_0^{2,2}(\Omega) \quad \text{as} \quad i\to\infty, \tag{6.5}$$

$$\chi_{\Omega\setminus\Omega_k(y_i)}\Delta y_i \to \Delta y \text{ strongly in } L^p(\Omega), \tag{6.6}$$

$$\lim_{i\to\infty}\int_\Omega \left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}} |\Delta y_i|^2 u_i\,dx = \int_\Omega |\Delta y|^p u\,dx, \tag{6.7}$$

where $\Omega_{k_i}(y_i)$ is defined by (6.2).

Proof. The proof is divided into five steps.

Step 1: $y_i \rightharpoonup y$ in $W_0^{2,2}(\Omega)$.- From Lemma 6.1 we deduce the existence of a subsequence, denoted in the same way $\{y_i\}_{i\in\mathbb{N}}\subset W_0^{2,2}(\Omega;\Gamma_D)$ and an element $y\in W_0^{2,p}(\Omega;\Gamma_D)$ such that $y_i\rightharpoonup y$ in $W_0^{2,2}(\Omega)$. Let us prove that y is the solution of (3.2)-(3.3). Let us fix an arbitrary test function $\varphi\in C_0^\infty(\Omega;\Gamma_D)$ and pass to the limit in the Minty inequality

$$\int_\Omega u_i(x)(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta\varphi|^2))^{\frac{p-2}{2}}\Delta\varphi\,(\Delta\varphi - \Delta y_i)\,dx \;\geqslant\; \int_\Omega f(\varphi - y_i)\,dx, \tag{6.8}$$

as $i \to \infty$. Taking into account that

$$
\begin{gathered}
(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta\varphi|^2))^{\frac{p-2}{2}}\Delta\varphi \to |\Delta\varphi|^{p-2}\Delta\varphi \ \text{ strongly in } L^r(\Omega), \text{ for all } 1 \leqslant r < \infty,\\
u_i \to u \ \text{ strongly in } L^r(\Omega), \text{ for all } 1 \leqslant r < \infty,\\
\Delta y_i \rightharpoonup \Delta y \ \text{ in } L^2(\Omega),
\end{gathered}
$$

we obtain

$$
\begin{gathered}
\lim_{i\to\infty}\int_\Omega (\varepsilon_i + \mathcal{F}_{k_i}(|\Delta\varphi|^2))^{\frac{p-2}{2}}|\Delta\varphi|^2 u_i\,dx = \int_\Omega |\Delta\varphi|^p u\,dx,\\
\lim_{i\to\infty}\int_\Omega (\varepsilon_i + \mathcal{F}_{k_i}(|\Delta\varphi|^2))^{\frac{p-2}{2}}\Delta\varphi\Delta y_i u_i\,dx = \int_\Omega |\Delta\varphi|^{p-2}\Delta\varphi\Delta y u\,dx.
\end{gathered}
$$

Thus, passing to the limit in relation (6.8) as $i \to \infty$, we arrive at the inequality (3.7) for every $\varphi \in C_0^\infty(\Omega;\Gamma_D)$. From density of $C_0^\infty(\Omega;\Gamma_D)$ in $W_0^{2,p}(\Omega;\Gamma_D)$, we infer that (3.7) holds for every $\varphi \in W_0^{2,p}(\Omega;\Gamma_D)$, and hence $y \in W_0^{2,p}(\Omega;\Gamma_D)$ is the solution to the boundary value problem (3.2)–(3.3) in the sense of distributions. Since the solution of (3.2)-(3.3) is unique, the whole sequence $\{y_i\}_{i\in\mathbb{N}}$ converges weakly to $y = y(u)$ in $W_0^{2,2}(\Omega)$.

Step 2: $\chi_{\Omega\setminus\Omega_k(y_i)}\Delta y_i \rightharpoonup \Delta y$ *in* $L^p(\Omega)$.- Following the definition of the sets $\Omega_{k_i}(y_i)$ and using (5.23), we obtain

$$
\begin{aligned}
\int_\Omega |\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i|^p\,dx &= \int_{\Omega\setminus\Omega_{k_i}(y_i)} |\Delta y_i|^p\,dx\\
&\leqslant \alpha^{-1}\int_{\Omega\setminus\Omega_{k_i}(y_i)} \left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}} |\Delta y_i|^2 u_i\,dx,\\
&\leqslant \alpha^{-1}\|y_i\|^p_{\varepsilon_i,k_i,u_i} \leqslant C < +\infty, \quad \forall\, i \in \mathbb{N}.
\end{aligned}
$$

Hence, taking a new subsequence if necessary, we infer the existence of a function $g \in L^p(\Omega)$ such that $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i \rightharpoonup g$ in $L^p(\Omega)$ as $i \to \infty$. Since $u_i \to u$ in $L^{p'}(\Omega)$, we conclude that

$$
\lim_{i\to\infty}\int_{\Omega\setminus\Omega_{k_i}(y_i)} \Delta y_i\varphi u_i\,dx = \int_\Omega g\varphi u\,dx, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{6.9}
$$

On the other hand, in view of the weak convergence $\Delta y_i \rightharpoonup \Delta y$ in $L^2(\Omega)$,

$$
\begin{aligned}
\int_\Omega \Delta y\varphi u\,dx &= \lim_{i\to\infty}\int_\Omega \Delta y_i\varphi u_i\,dx\\
&= \lim_{i\to\infty}\int_{\Omega\setminus\Omega_{k_i}(y_i)} \Delta y_i\varphi u_i\,dx + \lim_{i\to\infty}\int_{\Omega_{k_i}(y_i)} \Delta y_i\varphi u_i\,dx.
\end{aligned} \tag{6.10}
$$

Since

$$
\begin{aligned}
\left|\int_{\Omega_{k_i}(y_i)} \Delta y_i \varphi u_i \, dx\right| &\leqslant \|u_i\|_{L^\infty(\Omega)} \|\varphi\|_{C(\overline{\Omega})} \sqrt{|\Omega_{k_i}(y_i)|} \left(\int_{\Omega_{k_i}(y_i)} |\Delta y_i|^2 \, dx\right)^{1/2} \\
&\leqslant \frac{\|u_i\|_{L^\infty(\Omega)} \|\varphi\|_{C(\overline{\Omega})}}{\left(\varepsilon_i + k_i^2 + 1\right)^{\frac{p-2}{4}}} \sqrt{|\Omega_{k_i}(y_i)|} \|y_i\|^{\frac{p}{2}}_{\varepsilon_i, k_i, u_i} \\
&\overset{\text{by } (5.10),(5.23)}{\leqslant} \|\xi_2\|_{L^\infty(\Omega)} \|\varphi\|_{C(\overline{\Omega})} \frac{C}{k_i^{p-1}} \to 0 \quad \text{as} \quad i \to \infty,
\end{aligned}
$$

it follows from (6.9) and (6.10) that

$$
\int_\Omega g \varphi u \, dx = \int_\Omega \Delta y \varphi u \, dx, \quad \forall \varphi \in C_0^\infty(\Omega).
$$

Hence, $g = \Delta y$ almost everywhere in Ω and the convergence $\chi_{\Omega \setminus \Omega_k(y_i)} \Delta y_i \rightharpoonup \Delta y$ in $L^p(\Omega)$ holds.

Step 3: $\chi_{\Omega \setminus \Omega_k(y_i)} \Delta y_i \to \Delta y$ *in* $L^p(\Omega)$.- For each $i \in \mathbb{N}$, we have the energy equalities

$$
\begin{aligned}
\int_\Omega u_i (\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2))^{\frac{p-2}{2}} |\Delta y_i|^2 \, dx &= \int_\Omega f y_i \, dx, \\
\int_\Omega u(x) |\Delta y|^p \, dx &= \int_\Omega f y \, dx.
\end{aligned} \tag{6.11}
$$

From (6.11) and the fact that $y_i \rightharpoonup y$ in $W_0^{2,2}(\Omega)$, we deduce

$$
\begin{aligned}
\lim_{i \to \infty} \int_\Omega u_i (\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2))^{\frac{p-2}{2}} |\Delta y_i|^2 \, dx &= \lim_{i \to \infty} \left[\int_\Omega f y_i \, dx\right] \\
&= \int_\Omega f y \, dx \overset{\text{by } (6.11)_2}{=} \int_\Omega u |\Delta y|^p \, dx.
\end{aligned} \tag{6.12}
$$

Moreover, we have

$$
\begin{aligned}
&\int_\Omega u |\Delta y|^p \, dx = \lim_{i \to \infty} \int_\Omega \left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}} |\Delta y_i|^2 u_i \, dx \\
&\geqslant \limsup_{i \to \infty} \int_{\Omega \setminus \Omega_{k_i}(y_i)} \left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}} |\Delta y_i|^2 u_i \, dx \\
&\overset{\text{by } (5.7)}{\geqslant} \limsup_{i \to \infty} \int_{\Omega \setminus \Omega_{k_i}(y_i)} \left(\varepsilon_i + |\Delta y_i|^2\right)^{\frac{p-2}{2}} |\Delta y_i|^2 u_i \, dx \\
&\geqslant \limsup_{i \to \infty} \int_\Omega \chi_{\Omega \setminus \Omega_{k_i}(y_i)} |\Delta y_i|^p u_i \, dx \geqslant \liminf_{i \to \infty} \int_\Omega \chi_{\Omega \setminus \Omega_{k_i}(y_i)} |\Delta y_i|^p u_i \, dx.
\end{aligned} \tag{6.13}
$$

Since $u_i \to u$ in $L^r(\Omega)$ for every $1 \leqslant r < +\infty$, $\{u_i\}_i$ is bounded in $L^\infty(\Omega)$ and $u_i(x) \geqslant \alpha$ for almost all $x \in \Omega$, it is easy to check that $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i u_i^{1/p} \rightharpoonup \Delta y u^{1/p}$ in $L^p(\Omega)$. Using this convergence and (6.13) we get

$$\begin{aligned}&\int_\Omega u|\Delta y|^p\,dx \geqslant \limsup_{i\to\infty}\int_\Omega u_i\chi_{\Omega\setminus\Omega_{k_i}(y_i)}|\Delta y_i|^p\,dx\\ &\geqslant \liminf_{i\to\infty}\int_\Omega u_i\chi_{\Omega\setminus\Omega_{k_i}(y_i)}|\Delta y_i|^p\,dx = \liminf_{i\to\infty}\|\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i u_i^{1/p}\|^p_{L^p(\Omega)}\\ &\geqslant \|\Delta y u^{1/p}\|^p_{L^p(\Omega)} = \int_\Omega u|\Delta y|^p\,dx.\end{aligned}$$

The weak convergence $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i u_i^{1/p} \rightharpoonup \Delta y u^{1/p}$ in $L^p(\Omega)$ and the convergence of their norms $\|\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i u_i^{1/p}\|_{L^p(\Omega)} \to \|\Delta y u^{1/p}\|_{L^p(\Omega)}$ imply the strong convergence $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i u_i^{1/p} \to \Delta y u^{1/p}$ in $L^p(\Omega)$. Now, it is a simple exercise to check the strong convergence $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\Delta y_i \to \Delta y$ in $L^p(\Omega)$.

Step 4: Proof of (6.7).- From (6.6) and (6.13) we obtain

$$\lim_{i\to\infty}\int_{\Omega_{k_i}(y_i)}\left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}}|\Delta y_i|^2 u_i\,dx = 0. \tag{6.14}$$

Let us prove that

$$\lim_{i\to\infty}\int_{\Omega\setminus\Omega_{k_i}(y_i)}\left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}}|\Delta y_i|^2 u_i\,dx = \int_\Omega |\Delta y|^p u\,dx. \tag{6.15}$$

This is established as follows. From (5.7) we deduce

$$\begin{aligned}&\left(\varepsilon_i + \mathcal{F}_{k_i}(|\Delta y_i|^2)\right)^{\frac{p-2}{2}}|\Delta y_i|^2\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\\ &\qquad\leqslant (\varepsilon_i + \delta + |\Delta y_i|^2)^{\frac{p-2}{2}}|\Delta y_i|^2\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\\ &\qquad\leqslant 2^{\frac{p-2}{2}}((\varepsilon_i+\delta)^{\frac{p-2}{2}}|\Delta y_i|^2 + |\Delta y_i|^p)\chi_{\Omega\setminus\Omega_{k_i}(y_i)}.\end{aligned}$$

From (6.6) we know that the last term converges in $L^1(\Omega)$. Taking a subsequence if necessary we can dominate it by a $L^1(\Omega)$ function. Then by a simple application of Lebesgue's dominated convergence theorem we deduce (6.15). Finally, (6.14) and (6.15) imply (6.7).

Step 5: $y_i \to y$ in $W_0^{2,2}(\Omega)$.- First, we apply (6.14) to deduce

$$\lim_{i\to\infty}\int_{\Omega_k(y_i)}|\Delta y_i|^2\,dx \leqslant \frac{1}{\alpha}\lim_{i\to\infty}\int_{\Omega_k(y_i)}(\varepsilon_i + \mathcal{F}_k(|\Delta y_i|^2))^{\frac{p-2}{2}}|\Delta y_i|^2 u_i\,dx = 0.$$

Now, combining this estimate and (6.6) we conclude that

$$\Delta y_i = \chi_{\Omega_k(y_i)}\Delta y_i + \chi_{\Omega\setminus\Omega_k(y_i)}\Delta y_i \to \Delta y \ \text{ strongly in } \ L^2(\Omega).$$

$\square$

We are now in a position to show that optimal pairs to the approximated OCP $(\mathbb{P}_{\varepsilon,k})$ lead in the limit to optimal solutions of the original OCP $(\mathbb{P})$.

Theorem 6.1. *Let* $\left\{(u^0_{\varepsilon,k}, y^0_{\varepsilon,k})\right\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ *be an arbitrary sequence of optimal pairs to the approximating problems* $(\mathbb{P}_{\varepsilon,k})$. *Then, this sequence is bounded in* $BV(\Omega)\times W^{2,2}_0(\Omega)$ *and any cluster point* (u^0, y^0) *with respect to the (weak-*,weak) topology is a solution of the OCP* $(\mathbb{P})$. *Moreover, if for one subsequence we have* $u^0_{\varepsilon,k} \overset{*}{\rightharpoonup} u^0$ *in* $BV(\Omega)$ *and* $y^0_{\varepsilon,k} \rightharpoonup y^0$ *in* $W^{2,2}_0(\Omega)$, *then the following properties hold*

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} (u^0_{\varepsilon,k}, y^0_{\varepsilon,k}) = (u^0, y^0) \;\; \textit{strongly in} \;\; L^1(\Omega)\times W^{2,2}_0(\Omega), \tag{6.16}$$

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_\Omega |Du^0_{\varepsilon,k}| = \int_\Omega |Du^0|, \tag{6.17}$$

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})}\Delta y^0_{\varepsilon,k} = \Delta y^0 \;\; \textit{strongly in} \;\; L^p(\Omega), \tag{6.18}$$

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\Delta y^0_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\Delta y^0_{\varepsilon,k}|^2 u^0_{\varepsilon,k}\, dx = \int_\Omega |\Delta y^0|^p u^0\, dx, \tag{6.19}$$

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u^0_{\varepsilon,k}, y^0_{\varepsilon,k}) = I(u^0, y^0). \tag{6.20}$$

Proof. The boundedness of $\left\{(u^0_{\varepsilon,k}, y^0_{\varepsilon,k})\right\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ in $BV(\Omega)\times W^{2,2}_0(\Omega)$ is an immediate consequence of the boundedness of $\mathfrak{A}_{ad}$ in $BV(\Omega)$ and Lemma 6.1. Let us take a subsequence, denoted in the same way, such that $u^0_{\varepsilon,k} \overset{*}{\rightharpoonup} u^0$ in $BV(\Omega)$ and $y^0_{\varepsilon,k} \rightharpoonup y^0$ in $W^{2,2}_0(\Omega)$. From compactness property of BV-bounded sequences, we get that

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} u^0_{\varepsilon,k} = u^0 \;\; \text{strongly in} \;\; L^1(\Omega) \;\; \text{and} \quad \int_\Omega |Du^0| \leqslant \liminf_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_\Omega |Du^0_{\varepsilon,k}|. \tag{6.21}$$

From this convergence properties we infer that $u^0 \in \mathfrak{A}_{ad}$. Moreover, Lemma 6.2 implies that y^0 is the solution of (3.2)-(3.3) corresponding to $u = u^0$, therefore, in virew of (6.1), we deduce that $(u^0, y^0) \in \Xi$. Combining (6.5) and (6.21) we deduce (6.16). Convergences (6.18) and (6.19) follow from (6.6) and (6.7). Let us prove that (u^0, y^0) is a solution of $(\mathbb{P})$. Given an arbitrary element $(u, y) \in \Xi$, we define $u_{\varepsilon,k} = u$ and $y_{\varepsilon,k}$ as the solution of (5.2)-(5.3), hence $(u_{\varepsilon,k}, y_{\varepsilon,k}) \in \Xi_{\varepsilon,k}$. From (6.5) and (6.7) we get

$$I(u,y) = \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u, y_{\varepsilon,k}) = \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u_{\varepsilon,k}, y_{\varepsilon,k}).$$

Now, using (6.5), (6.16), (6.21), the above identity and the fact that $(u^0_{\varepsilon,k}, y^0_{\varepsilon,k})$ is a solution of $(\mathbb{P}_{\varepsilon,k})$, we get

$$\begin{aligned} I(u^0, y^0) &\leqslant \liminf_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u^0_{\varepsilon,k}, y^0_{\varepsilon,k}) \leqslant \limsup_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u^0_{\varepsilon,k}, y^0_{\varepsilon,k}) \\ &\leqslant \limsup_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u_{\varepsilon,k}, y_{\varepsilon,k}) = I(u, y). \end{aligned}$$

Since (u, y) is arbitrary in Ξ, this implies that (u^0, y^0) is a solution of $(\mathbb{P})$. Moreover, taking $(u, y) = (u^0, y^0)$ in the above inequalities, (6.20) is proved. Finally, (6.17) is an immediate consequence of (6.20) and the convergence properties established before. $\square$

Since Theorem 6.1 does not give an answer whether the entire set of solutions Ξ^{opt} to problem (3.2)–(3.6) can be attained in such a way, the following result shed some light on this matter.

Corollary 6.1. *Let $(u^0, y^0) \in \Xi^{opt}$ be an optimal solution to the OCP $(\mathbb{P})$ such that there is a closed neighborhood $\mathcal{U}(u^0)$ of u^0 in the norm topology of $L^1(\Omega)$ satisfying*

$$I(u^0, y^0) < I(u, y) \quad \forall\, u \in \mathfrak{A}_{ad} \cap \mathcal{U}(u^0) \text{ such that } (u, y) \in \Xi \text{ and } u \neq u^0. \quad (6.22)$$

Then there exists a sequence of local minima $(u^0_{\varepsilon,k}, y^0_{\varepsilon,k})$ of problems $(\mathbb{P}_{\varepsilon,k})$ such that

$$(u^0_{\varepsilon,k}, y^0_{\varepsilon,k}) \to (u^0, y^0) \quad \text{in the sense of Theorem 6.1.}$$

Proof. By the strict local optimality of (u^0, y^0), we have that it is the unique solution of

$$(\mathbb{Q}) \qquad \min_{(u,y)\in\Xi,\, u\in\mathcal{U}(u^0)} I(u, y).$$

For every ε and k let us consider the control problems

$$(\mathbb{Q}_{\varepsilon,k}) \qquad \min_{(u,y)\in\Xi_{\varepsilon,k},\, u\in\mathcal{U}(u^0)} I(u, y).$$

Since $(u^0, y_{\varepsilon,k}(u^0)) \in \Xi_{\varepsilon,k}$, it follows that $(\mathbb{Q}_{\varepsilon,k})$ has feasible controls, hence there exists at least one solution $(u^0_{\varepsilon,k}, y^0_{\varepsilon,k})$ of $(\mathbb{Q}_{\varepsilon,k})$ for every (ε, k). Now, arguing as in the proof of Theorem 6.1, we deduce that $(u^0_{\varepsilon,k}, y^0_{\varepsilon,k}) \to (\tilde{u}^0, \tilde{y}^0)$ strongly in $L^1(\Omega) \times W^{2,2}_0(\Omega)$, and $(\tilde{u}^0, \tilde{u}^0)$ is the unique solution of $(\mathbb{Q})$. This implies the existence of ε^0 and k^0 such that $u^0_{\varepsilon,k}$ belongs to the interior of $\mathcal{U}(u^0)$ for every $\varepsilon \leqslant \varepsilon^0$ and $k \geqslant k^0$. Consequently, $(u^0_{\varepsilon,k}, y^0_{\varepsilon,k})$ is a local minimum of $(\mathbb{P}_{\varepsilon,k})$ for every $\varepsilon \leqslant \varepsilon^0$ and $k \geqslant k^0$. This concludes the proof. $\square$

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