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Проблеми математичного моделювання
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ON AN APPROACH TO SOLVE THE NONLINEAR ELLIPTIC EQUATIONS OF VON KARMAN TYPE

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The present paper deals with behaviour of approximate solution sequence of boundary problems for the nonlinear elliptic equations of von Karman type constructed with the employment of the iterative generalized Kantorovich method; the relations for the generalized solution of equations in question is used as the governing ones. We revealed correctness of the relations as well as strongly continuity of the method operator in specific 'weighted' space that makes it possible to state sufficient conditions for strong convergence.

Keywords: elliptic equations of von Karman type, the iterative generalized Kantorovich method, generalized solution.

1. Introduction

A wealth of models (with a great number of applications in nano-, bio- and, space technologies) are described with the employment of boundary problems for nonlinear elliptic equations of von Karman type. Since the equations in question are strongly non-linear and their solutions observed experimentally are rather variegated, it is necessary to develop a numerical method able to construct solutions of PDE of this class, locate its singular points and (for bifurcation points) trace the respective bifurcation paths. Notwithstanding the finite element method is popular in actual practice, attempts to utiltize it to analyze non-linear boundary problems for von Karman equations encounters with difficulties [11]. This makes explicable constant interest to meshfree (meshless) numerical algorithms and to their applications to the non-linear problem under study. A review [9] discuses a group of method based upon functions with bounded definitional domain that used to approximate unknown functions in the neighbourhood of discretization nodes; similar concepts are introduced in non-parametric regression [10] and artificial neural networks using the radial basis functions [3]. Another group of methods employed to solve non-linear boundary problems for partial differential equations is associated with an adaptive basis; as opposed to conventional numerical methods that imply that a basis is fixed and does not depend on features of the problem to be solved (one can vary only basis coefficients only), these methods adjust basis functions according to the problem at hand. For example, the paper [2] considers adaptive wavelet collocation; the paper [6] deals with its modification grounded upon second generation wavelets. This approach combines adaptability characteristic to the methods using adaptive wavelets with flexibility of collocation methods. In such methods, one employs wavelets to adjust meshing and finite differences to approximate partial derivations. Alternative approach to construct an adaptive optimal basis is traced back to the generalized Kantorovich method [4, 5]. It implies that unknown functions are presented as a sum of products of single-variable unknown functions to solve a sequence of non-linear boundary problems for ODEs in order to solve the initial non-linear boundary for PDEs. The paper [1] applies the method under discussion to a linear boundary problem, allowing close solution due to its linearity. If the methods is applied to non-linear boundary problem (such as von Karman-type one), then it is necessary to establish iterative process [12] that incorporates algorithms used to solve non-linear boundary problems for ODEs; such method is named the iterative generalized Kantorovich method (IGKM). The present paper examines applicability of IGKM to non-linear elliptic of von Karman-type equations.

2. The generalized solution of von Karman-type equations

Non-linear elliptic von Karman-type equations (in orthogonal coordinate system (x_1, x_2)) reads as the following system of PDEs

$$a_1 \nabla^{2m} u_1 + T(u_1, u_2) - \nabla_k^2 u_2 = q, \tag{2.1}$$

$$a_2 \nabla^{2m} u_2 - \frac{1}{2} T(u_1, u_1) - \nabla_k^2 u_1 = 0,$$

where $q = q(x_1, x_2)$ is the function describing external influence;

$$\nabla^{2m} \alpha = \sum_{i=0}^{m} C_m^i \frac{\partial^{2m} \alpha}{\partial x_1^{2i} \partial x_2^{2m-2i}};$$

$$\nabla_k^2 \alpha = k_1 \frac{\partial^2 \alpha}{\partial x_1^2} + k_2 \frac{\partial^2 \alpha}{\partial x_2^2};$$

$$m \in N_+;$$

$$T(\alpha, \beta) = \sum_{p=1}^{P} c_p \left(T_p^{(1)}(\alpha, \beta) + T_p^{(2)}(\alpha, \beta) \right); \tag{2.2}$$

$$T_p^{(1)}(\alpha,\beta) = (-1)^{s_p} \sum_{j=0}^{s_p^{(1)}} \sum_{i=0}^{s_p^{(2)}} C^i_{s_p^{(2)}} C^j_{s_p^{(1)}} \frac{\partial^{q_p+s_p-i-j}\alpha}{\partial^{q_p^{(1)}+s_p^{(1)}-j}x_1 \partial^{q_p^{(2)}+s_p^{(2)}-i}x_2} \frac{\partial^{t_p+i+j}\beta}{\partial^{t_p^{(1)}+j}x_1 \partial^{t_p^{(2)}+i}x_2}$$

$$T_p^{(2)}(\alpha,\beta) = (-1)^{q_p} \sum_{j=0}^{q_p^{(1)}} \sum_{i=0}^{q_p^{(2)}} C^i_{q_p^{(2)}} C^j_{q_p^{(1)}} \frac{\partial^{q_p+s_p-i-j}\alpha}{\partial^{q_p^{(1)}+s_p^{(1)}-j}x_1 \partial^{q_p^{(2)}+s_p^{(2)}-i}x_2} \frac{\partial^{t_p+i+j}\beta}{\partial^{t_p^{(1)}+j}x_1 \partial^{t_p^{(2)}+i}x_2}$$

$$0 < s_p, t_p < 2m; s_p^{(1)} + s_p^{(2)} = s_p, t_p^{(1)} + t_p^{(2)} = t_p;$$

k_1, k_2, a_1, a_2, c_p are positive constants.

The problem is defined on the domain

$$\Omega = \left\{x_1^{\min} \leqslant x_1 \leqslant x_1^{\max}; x_2^{\min} \leqslant x_2 \leqslant x_2^{\max}\right\},$$

bounded by

$$\Gamma \equiv \partial\Omega = \left\{x_1 = x_1^{\min}, x_2^{\min} \leqslant x_2 \leqslant x_2^{\max}\right\} \bigcup \left\{x_1 = x_1^{\max}, x_2^{\min} \leqslant x_2 \leqslant x_2^{\max}\right\}.$$

The operator of boundary conditions, defined on Γ is denoted as

$$S\left(u_1\left|_\Gamma\right., \ u_2\left|_\Gamma\right.\right) = 0. \tag{2.3}$$

$\overset{0}{C}{}^{(m)}(\Omega)$ symbolizes a set of continuous (with its derivatives up to order m) functions, which satisfy the homogeneous boundary conditions $\left\{\alpha\left|_\Gamma\right. = 0, \frac{\partial\alpha}{\partial n}\left|_\Gamma\right. = 0, \ldots, \frac{\partial^{m-1}\alpha}{\partial n^{m-1}}\left|_\Gamma\right. = 0\right\}$. Then $H_0(\Omega)$ is the Sobolev space due to closure of $\overset{0}{C}{}^{(m)}(\Omega)$ in the norm $\iint_\Omega K(\alpha,\alpha)\,dx_1dx_2 \equiv \|\alpha\|_H^2$, where

$$K(\alpha,\beta) = \sum_{\substack{p:\, m_1^{(p)}+m_2^{(p)}=m \\ q:\, m_1^{(q)}+m_2^{(q)}=m}} b_{pq} \left[\frac{\partial^m\alpha}{\partial x_1^{m_1^{(p)}}\partial x_2^{m_2^{(p)}}}\frac{\partial^m\beta}{\partial x_1^{m_1^{(q)}}\partial x_2^{m_2^{(q)}}} + \frac{\partial^m\alpha}{\partial x_1^{m_1^{(q)}}\partial x_2^{m_2^{(q)}}}\frac{\partial^m\beta}{\partial x_1^{m_1^{(p)}}\partial x_2^{m_2^{(p)}}}\right], \tag{2.4}$$

$$2(-1)^m \sum_{\substack{p:\, m_1^{(p)}+m_1^{(q)}=2i \\ q:\, m_2^{(p)}+m_2^{(q)}=2m-2i}} b_{pq} = C_m^i, i = \overline{0,m}, \tag{2.5}$$

$$\sum_{\substack{p:\, m_1^{(p)}+m_1^{(q)}=j_1\neq 2i \\ q:\, m_2^{(p)}+m_2^{(q)}=j_2\neq 2m-2i}} b_{pq}=0,$$

$$\operatorname{sign} b_{pq}=(-1)^m .$$

The expression $\iint_\Omega K(\alpha,\beta)\,dx_1dx_2$ meets all demands on scalar products defined in Hilbert space. The sole fact to be proved is

Lemma 2.1. *If* $\|\alpha\|_{H_0(\Omega)}=0$ *then* $\alpha\equiv 0$ *in* $\bar{\Omega}$.

Proof. Constraints (2.5) ensure in particular that α :

$$\|\alpha\|_{H(\Omega)}=0 \;\text{ for all }\; \iint_\Omega \left(\frac{\partial^m \alpha}{\partial x_1^{m_1}\partial x_2^{m_2}}\right)^2 dx_1dx_2=0.$$

Therefore, if one considers the boundary conditions for functions belonging to $\overset{0}{C}{}^{(m)}(\Omega)$, one finds that $\alpha\equiv 0$ in $\bar{\Omega}$. □

The space H is a closure of a set of functions $C^{(m)}(\Omega)$ in the same norm, while boundary conditions defined on Γ are assumed to guarantee that Lemma 2.1 holds true.

The generalized solution of the non-linear boundary problem in question is a pair of function $U=(u_1,\ u_2)$, $u_1\in W_2^{(m)}(\Omega)$, $u_2\in H_0(\Omega)$, satisfying the integral equations

$$a_1\iint_\Omega K(u_1,v_1)\,dx_1dx_2=\iint_\Omega \left[\nabla_k^2 u_2 v_1 - Q_1(u_1,u_2,v_1)-qv_1\right]dx_1dx_2, \quad (2.6)$$

$$a_2\iint_\Omega K(u_2,v_2)\,dx_1dx_2=\iint_\Omega \left[\nabla_k^2 v_2 u_1 - Q_2(u_1,v_2)\right]dx_1dx_2$$

for arbitrary functions $v_1\in W_2^{(m)}(\Omega)$, $v_2\in H_0(\Omega)$.

Expressions (2.6) use the following notations

$$Q_1(\alpha,\beta,\gamma)=\sum_{p=1}^{P} c_p$$

$$\left(\frac{\partial^{s_p}\alpha}{\partial x_1^{s_p^{(1)}}\partial x_2^{s_p^{(2)}}}\,\frac{\partial^{t_p}\beta}{\partial x_1^{t_p^{(1)}}\partial x_2^{t_p^{(2)}}}\,\frac{\partial^{q_p}\gamma}{\partial x_1^{q_p^{(1)}}\partial x_2^{q_p^{(2)}}}+\frac{\partial^{q_p}\alpha}{\partial x_1^{q_p^{(1)}}\partial x_2^{q_p^{(2)}}}\,\frac{\partial^{t_p}\beta}{\partial x_1^{t_p^{(1)}}\partial x_2^{t_p^{(2)}}}\,\frac{\partial^{s_p}\gamma}{\partial x_1^{s_p^{(1)}}\partial x_2^{s_p^{(2)}}}\right);$$

$$Q_2(\alpha,\beta)=\sum_{p=1}^{P}c_p\frac{\partial^{s_p}\alpha}{\partial x_1^{s_p^{(1)}}\partial x_2^{s_p^{(2)}}}\frac{\partial^{q_p}\alpha}{\partial x_1^{q_p^{(1)}}\partial x_2^{q_p^{(2)}}}\frac{\partial^{t_p}\beta}{\partial x_1^{t_p^{(1)}}\partial x_2^{t_p^{(2)}}}; \tag{2.7}$$

$$a_p \in R, 0 \leqslant s_p, q_p, t_p < m.$$

The immediate corollary of the Sobolev embedding theorem is [13], [15]:

Theorem 2.1. *If $\alpha \in W_2^{(m)}(\Omega)$, then $\frac{\partial^p \alpha}{\partial x_1^{m_1}\partial x_2^{m_2}} \in L_q(\Omega)$, $m_1+m_2=p$, $1\leqslant p\leqslant m-1$, for any $q\geqslant 1$. In doing so the embedding operator is strongly continuous and thus weak convergence $\alpha_n \to \alpha_0$ in $W_2^{(m)}(\Omega)$ guarantees strong convergence $\frac{\partial^p \alpha_n}{\partial x_1^{m_1}\partial x_2^{m_2}} \Rightarrow \frac{\partial^p \alpha_0}{\partial x_1^{m_1}\partial x_2^{m_2}}$ in $L_q(\Omega)$ for any $q\geqslant 1$. Furthermore,*

$$\left\|\frac{\partial^p \alpha}{\partial x_1^{m_1}\partial x_2^{m_2}}\right\|_{L_q(\Omega)} \leqslant c\,\|\alpha\|_{W_2^{(m)}(\Omega)}. \tag{2.8}$$

Another theorem of consequences [15] is a theorem that states sufficient conditions of functional coercitivity.

Theorem 2.2. *Let two-dimensional plane region Ω be that of the Sobolev class $(2,\ m,\ 2)$. Let quadratic functional $W_2^{(m)}(\Omega)$ be as follows*

$$\Re(U)=\iint_\Omega\left[\Re_2\left(\frac{\partial^m U}{\partial x_1^{m_1}\partial x_2^{m_2}}\right)+\Re_1\left(\frac{\partial^m U}{\partial x_1^{m_1}\partial x_2^{m_2}}\right)\right]d\Omega+\Re_0(U), \tag{2.9}$$

where U is a two-dimensional vector; $\Re_2$ is a quadratic form with coefficients continuous in Ω; $\Re_1$ is a linear form of derivatives of order m with coefficients that are linear form of derivatives of order less than m, and $\Re_1$ thereby is a quadratic functional in $W_2^{(m)}(\Omega)$. Moreover, let $\Re_0(U)$ be weakly continuous in $W_2^{(m)}(\Omega)$. Finally, let the following conditions be true:

1. *$\Re(U)\geqslant 0$ and $\Re(U)=0$ results in $U\equiv 0$ in Ω;*
2. *if $U_n \to 0$ in $W_2^{(m)}(\Omega)$ and $\Re(U_n)\to 0$, then $\left\|\frac{\partial^m U}{\partial x_1^{m_1}\partial x_2^{m_2}}\right\|_{L_2(\Omega)} \to 0$, for arbitrary $m_1+m_2=m$.*

In that case

$$\Re(U)\geqslant m\,\|U\|^2_{W_2^{(m)}(\Omega)}. \tag{2.10}$$

Theorem 2.3. *For all elements $\alpha\in W_2^{(m)}(\Omega)$ the following inequality holds true*

$$\|\alpha\|_H \geqslant m\,\|\alpha\|_{W_2^{(m)}(\Omega)}. \tag{2.11}$$

Proof. Theorem 2.2 is employed to prove that statement. Let $\Re(\alpha) = \|\alpha\|_H^2$ be with $\Re_1(\alpha) \equiv \Re_0(\alpha) \equiv 0$. The condition 1 is a direct sequence of the Lemma 2.1. To verify the condition 2 let us assume that $\alpha_n \to 0$ in $W_2^{(m)}(\Omega)$ and $\|\alpha_n\|_H \to 0$. In virtue of (2.5), expression $\|\alpha\|_H^2$ can be reduced to

$$K(\alpha, \beta) = 2 \sum_{p:m_1^{(p)}+m_2^{(p)}=m} \bar{b}_{pq} \left\| \frac{\partial^m \alpha}{\partial x_1^{m_1^{(p)}} \partial x_2^{m_2^{(p)}}} \right\|_{L_2(\Omega)}^2 ,$$

$\bar{b}_{pq} > 0$.The last proves the condition 2 and thereby the theorem. □

Theorem 2.4. *Spaces* $H(\Omega)(H_0(\Omega))$ *and* $W_2^{(m)}(\Omega)(\overset{0}{W}{}_2^{(m)}(\Omega))$ *are equivalent and contains the same elements.*

Proof. An obvious inequality

$$\|\alpha\|_H \leqslant m \|\alpha\|_{W_2^{(m)}(\Omega)} \tag{2.12}$$

and the inequality (2.11) proved by the theorem 2.3 insure that the two norms are equivalent. As far as both spaces are closures of the same functional space $C^{(m)}(\Omega)(\overset{0}{C}{}^{(m)}(\Omega))$ in the equivalent norms, they coincide. □

Theorem 4 proves the important statement: For all elements of H the embedding theorem 1 holds true:

Theorem 2.5. *If* $\alpha \in H$ *then* $\frac{\partial^p \alpha}{\partial x_1^{m_1} \partial x_2^{m_2}} \in L_q(\Omega)$, $m_1 + m_2 = p$, $1 \leqslant p \leqslant m-1$ *for any* $q \geqslant 1$. *Moreover, the embedding operator is strongly continuous and thus a weak convergence* $\alpha_n \to \alpha_0$ *in* $H(\Omega)$ *guarantees a strong convergence* $\frac{\partial^p \alpha_n}{\partial x_1^{m_1} \partial x_2^{m_2}} \Rightarrow \frac{\partial^p \alpha_0}{\partial x_1^{m_1} \partial x_2^{m_2}}$ *in* $L_q(\Omega)$ *for any* $q \geqslant 1$.

Besides that

$$\left\| \frac{\partial^p \alpha}{\partial x_1^{m_1} \partial x_2^{m_2}} \right\|_{L_q(\Omega)} \leqslant c \|\alpha\|_{H(\Omega)} . \tag{2.13}$$

Similar theorem is valid for elements of H_0.

Theorem 2.6. *Let*

1. *A definitional domain of the non-linear boundary problem* (2.1), (2.3) *is a smooth manifold of the class* $C^{(2)}(\Omega)$, *where* Ω *is a Sobolev region of the class* $(2,\ m,\ 2)$.
2. *The respective boundary* Γ *is piecewise-continuous (of the class* $C_\Gamma^{(1)}$*).*

3. Boundary conditions on Γ (2.3) *allow constructing the space* H.

Then the definition of the generalized solution is correct that is each right-hand term of the integral relations (2.6) *is determined if* $u_1 \in W_2^{(m)}(\Omega)$, $u_2 \in H_0(\Omega)$. *Furthermore, these terms are linear and bounded functionals of* v_1, v_2 *in* $H(\Omega), H_0(\Omega)$.

Proof. If $u_1, v_1 \in W_2^{(m)}$, $u_2, v_2 \in H_0$ then (due to the theorems 2.1 and 2.4) all co-products defining the right-hand terms (2.6) belong to the space H. Besides that,in virtue of the theorems indicated the following is true:

$$\left|\iint_\Omega \nabla_k^2 u_2 v_1 dx_1 dx_2\right| \leqslant \left\|\nabla_k^2 u_2\right\|_{L_2} \|v_1\|_{L_2} \leqslant \left(\sum_{i=1}^2 |k_i| \left\|\frac{\partial^2 u_2}{\partial x_i^2}\right\|_{L_2}\right) \|v_1\|_{L_2} \leqslant m \sum_{i=1}^2 |k_i| \, \|u_2\|_H \|v_1\|_H$$

(in virtue of the Theorem 2.5 and of the sequence of inequalities [due to the Theorem 2.4] $\|v_1\|_{L_2} \leqslant \|v_1\|_{W_2^{(m)}} \leqslant m\|v_1\|_H$).

Analogously, the following inequality holds true for the second term

$$\iint_\Omega Q_1 dx_1 dx_2 \leqslant \sum_{p=0}^P c_p\Bigg(\left\|\frac{\partial^{s_p} u_1}{\partial x_1^{s_p^{(1)}} \partial x_2^{s_p^{(2)}}}\right\|_{L_4} \left\|\frac{\partial^{t_p} u_2}{\partial x_1^{t_p^{(1)}} \partial x_2^{t_p^{(2)}}}\right\|_{L_4} \left\|\frac{\partial^{q_p} v_1}{\partial x_1^{q_p^{(1)}} \partial x_2^{q_p^{(2)}}}\right\|_{L_2} + \left\|\frac{\partial^{q_p} u_1}{\partial x_1^{q_p^{(1)}} \partial x_2^{q_p^{(2)}}}\right\|_{L_4} \left\|\frac{\partial^{t_p} u_2}{\partial x_1^{t_p^{(1)}} \partial x_2^{t_p^{(2)}}}\right\|_{L_4} \left\|\frac{\partial^{s_p} v_1}{\partial x_1^{s_p^{(1)}} \partial x_2^{s_p^{(2)}}}\right\|_{L_2} \Bigg) \leqslant 2m \sum_{p=1}^P |a_p| \, \|u_1\|_H \|u_2\|_H \|v_1\|_H ;$$

and for the third one:

$$\left|\iint_\Omega q v_1 dx_1 dx_2\right| \leqslant m\|q\|_{L_2} \|v_1\|_H ,$$

This proves the theorem with respect to (2.4).

For the terms of the second equation, the following is valid

$$\left|\iint_\Omega \nabla_k^2 v_2 u_1 dx_1 dx_2\right| \leqslant \left\|\nabla_k^2 v_2\right\|_{L_2} \|u_1\|_{L_2} \leqslant \left(\sum_{i=1}^2 |k_i| \left\|\frac{\partial^2 v_2}{\partial x_i^2}\right\|_{L_2}\right) \|u_1\|_{L_2} \leqslant m \sum_{i=1}^2 |k_i| \, \|u_1\|_H \|v_2\|_H ;$$

$$\iint_{\Omega} Q_2 dx_1 dx_2 \leqslant \sum_{p=0}^{P} c_p \left\| \frac{\partial^{s_p} u_1}{\partial x_1^{s_p^{(1)}} \partial x_2^{s_p^{(2)}}} \right\|_{L_4} \left\| \frac{\partial^{q_p} u_1}{\partial x_1^{q_p^{(1)}} \partial x_2^{q_p^{(2)}}} \right\|_{L_4} \left\| \frac{\partial^{t_p} v_2}{\partial x_1^{t_p^{(1)}} \partial x_2^{t_p^{(2)}}} \right\|_{L_2}$$
$$\leqslant 2 \sum_{p=1}^{P} c_p \left\| u_1 \right\|_H^2 \left\| v_2 \right\|_H .$$

□

3. The iterative generalized Kantorovich method

To construct solutions of the non-linear boundary problem (2.1), (2.3), given by relations of its generalized solution (2.6), approximate sequence is generated with the employment of the representation the vector of unknown functions

$$u_j^{(i)}(x_1, x_2) = h_{\rm j}^{(\rm i)}(x_1)\, \bar{g}_j^{(i)}(x_2) + \bar{h}_{\rm j}^{(\rm i)}(x_1)\, g_j^{(i)}(x_2), j = \overline{1,2}, i \to \infty,$$

$$h_1^{(i)}(x_1)\, \bar{g}_1^{(i)}(x_2) + \bar{h}_1^{(i)}(x_1)\, g_1^{(i)}(x_2) \in H(\Omega), \tag{3.1}$$

$$h_2^{(i)}(x_1)\, \bar{g}_2^{(i)}(x_2) + \bar{h}_2^{(i)}(x_1)\, g_2^{(i)}(x_2) \in H_0(\Omega)$$

on the iterations of the method. The representation considered is grounded on the analogies with [14], [7], [8]. Each iteration comprises calculation both $h_{\rm j}^{(\rm i)}(x_1)$ and $g_j^{(i)}(x_2)$. The functions with overbars are known from the previous iteration.

The representation (3.1) implies that a solution of the non-linear boundary problem (2.1), (2.3) is sought as the sequence of solutions for ordinary differential equations

$$\frac{dh}{dx_1} = f^{x_1}\left(h(x_1), \bar{h}(x_1), a^{x_2}, q\right), x_1 \in \left[x_1^{\min}; x_1^{\max}\right]; \tag{3.2}$$

$$\frac{dg}{dx_2} = f^{x_2}\left(g(x_2), \bar{g}(x_2), a^{x_1}, q\right), x_1 \in \left[x_2^{\min}; x_2^{\max}\right]. \tag{3.3}$$

For the sake of brevity, the superscripts corresponding to iteration are omitted in (3.2), (3.3). The vectors $h(x_1)$ and $g(x_2)$ are $h(x_1) = \left\{\frac{d^s h_1(x_1)}{dx_1^s}, \frac{d^s h_2(x_1)}{dx_1^s}\right\}$, $g(x_1) = \left\{\frac{d^s g_1(x_2)}{dx_2^s}, \frac{d^s g_2(x_2)}{dx_2^s}\right\}$, $s = \overline{0,3}$; $\bar{h}(x_1) = h^{(i-1)}(x_1)$, $\bar{g}(x_2) = g^{(i-1)}(x_2)$.

Elements of the vectors a^{x_1}, a^{x_2} are definite integrals of components of $h(x_1)$, $\bar{h}(x_1)$ and $g(x_2)$, $\bar{g}(x_2)$, respectively. If the components of $g(x_2)$ are calculated first then components of $h(x_1)$ are replaced by the appropriate components of $\bar{h}(x_1)$ while the vector a^{x_1} is formed.

It is worth stressing that an order of the equations (3.2), (3.3) does not depend on the way the initial PDEs (2.1) are approximated, but does depend only on their order. Ordinary differential equations (3.2), (3.3) must be completed by point-wise boundary conditions derived from boundary conditions (2.3).

IGKM refers to an iterative process such that the subsystems (3.2), (3.3) are resolved separately and consequently; therefore solution of the non-linear boundary problem for PDEs is found using solutions for the sequence of boundary problems for ODEs. The process is completed, if the norms of solutions differences for several consequent iterations are small.

4. Investigation of convergence

Linearity and boundness of the functional $\iint_\Omega qv_1 dx_1 dx_2$ allows its representation (due to the Riesz theorem) $\iint_\Omega qv_1 dx_1 dx_2 = (u^*, v_1)_H$, and, in turn, (owing to $u_1 \to u_1 + u^*$) representation of the generalized solution

$$a_1 \iint_\Omega K(u_1, v_1)\, dx_1 dx_2 = -\, a_1 \iint_\Omega K(u^*, v_1)\, dx_1 dx_2 + \iint_\Omega \left[\nabla_k^2 u_2 v_1 - Q_1(u_1 + u^*, u_2, v_1)\right] dx_1 dx_2, \tag{4.1}$$

$$a_2 \iint_\Omega K(u_2, v_2)\, dx_1 dx_2 = \iint_\Omega \left[\nabla_k^2 v_2 (u_1 + u^*) - Q_2(u_1 + u^*, v_2)\right] dx_1 dx_2 \tag{4.2}$$

The representation (3.1) makes it necessary to study specific 'weighted' functional spaces H_g, $g \in W_2^{(m)}([c; d])$, defined as a closure of a set of functions $W_2^{(m)}([a; b])$ in the norm

$$\|h\|_{H_g}^2 = \sum_{\substack{p:\, m_1^{(p)} + m_2^{(p)} = m \\ q:\, m_1^{(q)} + m_2^{(q)} = m}} d_{pq} \int h^{\left(m_1^{(p)}\right)} h^{\left(m_1^{(q)}\right)} dx_1, \tag{4.3}$$

where $d_{pq} = 2b_{pq} \int g^{\left(m_2^{(p)}\right)} g^{\left(m_2^{(q)}\right)} dx_2$.

The obvious equality $\|hg\|_H = \|h\|_{H_g} = \|g\|_{H_h}$ is true.

Lemma 4.1. *Spaces $H_g([a, b])$ and $W_2^{(m)}([a, b])$ and contains the same elements (if the function g is not the null equation).*

Proof. The proof is carried out along similar lines with that of the theorem 2.4. □

Lemma 2.2 leads to the following lemma:

Lemma 4.2. *Let approximate sequences of the iterative generalized Kantorovich method* $\left\{g_1^{(i)}\right\}$, $\left\{h_1^{(i)}\right\}$ *belong to bounded domains of* $W_2^{(m)}\left(\left[x_1^{\min};x_1^{\max}\right]\right)$ *and* $W_2^{(m)}\left(\left[x_2^{\min};x_2^{\max}\right]\right)$*, respectively. Then the spaces* $H_g\left([a,b]\right)$ *and* $H_{g_i}\left([a,b]\right)$ *are equivalent and contain the same elements.*

Proof. The proof is carried out along similar lines with that of the theorem 2.4. □

In the framework of the considered representation, the integral relations (4.1), (4.2) on the iterations of the method are equivalent to two pairs of integral equations for $g^{(i)}$ and $h^{(i)}$, respectively.

$$\begin{aligned} a_1 \iint_\Omega K\left(\bar{h}_1 g_1, \bar{h}_1\varphi_1\right) dx_1dx_2 = -a_1 \iint_\Omega K\left(h_1\bar{g}_1, \bar{h}_1\varphi_1\right) dx_1dx_2 \\ - a_1 \iint_\Omega K\left(h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, \bar{h}_1\varphi_1\right) dx_1dx_2 + \iint_\Omega \nabla_k^2\left[h_2\bar{g}_2 + \bar{h}_2 g_2\right]\bar{h}_1\varphi_1 dx_1dx_2 \\ - \iint_\Omega Q_1\left(h_1\bar{g}_1 + \bar{h}_1 g_1 + h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, h_2\bar{g}_2 + \bar{h}_2 g_2, \bar{h}_1\varphi_1\right) dx_1dx_2, \end{aligned} \tag{4.4}$$

$$\begin{aligned} a_2 \iint_\Omega K\left(\bar{h}_2 g_2, \bar{h}_2\varphi_2\right) dx_1dx_2 = -a_2 \iint_\Omega K\left(h_2\bar{g}_2, \bar{h}_2\varphi_2\right) dx_1dx_2 \\ + \iint_\Omega \nabla_k^2\left[\bar{h}_2\varphi_2\right]\left(h_1\bar{g}_1 + \bar{h}_1 g + h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*\right) dx_1dx_2 \\ - \iint_\Omega Q_2\left(h_1\bar{g}_1 + \bar{h}_1 g + h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, \bar{h}_2\varphi_2\right) dx_1dx_2 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} a_1 \iint_\Omega K\left(\bar{h}_1 g_1, \psi_1\bar{g}_1\right) dx_1dx_2 = -a_1 \iint_\Omega K\left(h_1\bar{g}_1, \psi_1\bar{g}_1\right) dx_1dx_2 \\ - a_1 \iint_\Omega K\left(h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, \psi_1\bar{g}_1\right) dx_1dx_2 + \iint_\Omega \nabla_k^2\left[h_2\bar{g}_2 + \bar{h}_2 g_2\right]\psi_1\bar{g}_1 dx_1dx_2 \\ - \iint_\Omega Q_1\left(h_1\bar{g}_1 + \bar{h}_1 g_1 + h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, h_2\bar{g}_2 + \bar{h}_2 g_2, \psi_1\bar{g}_1\right) dx_1dx_2, \end{aligned} \tag{4.6}$$

$$\begin{aligned} a_2 \iint_\Omega K\left(\bar{h}_2 g_2, \psi_2\bar{g}_2\right) dx_1dx_2 = -a_2 \iint_\Omega K\left(h_2\bar{g}_2, \psi_2\bar{g}_2\right) dx_1dx_2 \\ + \iint_\Omega \nabla_k^2\left[\psi_2\bar{g}_2\right]\left(h_1\bar{g}_1 + \bar{h}_1 g + h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*\right) dx_1dx_2 \\ - \iint_\Omega Q_2\left(h_1\bar{g}_1 + \bar{h}_1 g + h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, \psi_2\bar{g}_2\right) dx_1dx_2. \end{aligned} \tag{4.7}$$

In the expressions(4.4)–(4.7), for the sake of brevity, the superscript corresponding to iterations is omitted. One assumes that arbitrary functions v_j are presented as $u_{\rm j}(x_1, x_2) = \psi_j(x_1)\,\bar{g}_j^{(i)}(x_2) + \bar{h}_{\rm j}^{(\rm i)}(x_1)\,\varphi_j(x_2)\,,\; j = \overline{1,2}\,,\; i \to \infty$.

When h is replaced by $\bar{h}$ in (4.4), (4.5), in compliance with IGKM procedure, one can consider the left-hand side of (4.4) as a linear functional of φ_2 in $H_{\bar{h}_2^{(i)}}$. Consequently, the right-hand one can also be presented as
$\left(P^{(i)}\left(g_1^{(i)}\right), \varphi_2\right)_{H_{\bar{h}_2^{(i)}}}$,(due to the Riesz theorem), that leads to

$$g_2^{(i)} = P^{(i)}\left(g_1^{(i)}\right). \tag{4.8}$$

Here, the superscript i in the notation $P^{(i)}$ signifies that its definition depends on definite integrals of functions $h_j^{(i-1)}$ calculated on the previous iteration of IGKM.

Substituting relations (4.8) into the right-hand side of (4.5), one obtains a linear functional of φ_1 in $H_{h_1^{(i-1)}}$, which can be presented (due to the Riesz theorem) as $\left(G^{(i)}\left(g_1^{(i)}\right), \varphi_1\right)_{H_{\bar{h}_1^{(i)}}}$ likewise.

Since φ_1 is arbitrary, the resultant expression is

$$g_1^{(i)} = G^{(i)}\left(g_1^{(i)}, g_2^{(i)}\right). \tag{4.9}$$

Combined expressions (4.8) and (4.9) form the operator representation of the governing equations used to determine the vector-function g in the frameworks of IGKM:

$$g^{(i)} = A_{x_2}^{(i)}\left(g^{(i)}\right) \tag{4.10}$$

The operators $N^{(i)}$, $H^{(i)}$, and $A_{x_1}^{(i)}$ used to determine the vector-function h are introduced along similar lines

$$h^{(i)} = A_{x_2}^{(i)}\left(h^{(i)}\right) \tag{4.11}$$

In these terms, IGKM is considered to be a sequence of operator equations defined using operators A_{x_1} and A_{x_2} such that their action for the i-th iteration coincide with the action of $A_{x_1}^{(i)}$ and $A_{x_2}^{(i)}$.

Theorem 4.1. *It is necessary and sufficient for h_1, h_2, $\bar{h}_1$, $\bar{h}_2$, g_1, g_2, $\bar{g}_1$, $\bar{g}_2$ to satisfy the generalized solution of the problem* (2.1), (2.3), *taking into account the representation* (3.1), *to satisfy the operator equations* (4.10) *and* (4.11).

Theorem 4.2. *The following representations hold true:*

$$P = P_0 + P_1 + P_2, \tag{4.12}$$

$$G = G_0 + G_1 + G_2 + G_3, \tag{4.13}$$

where P_s, G_s are homogeneous operators of order s.

Proof. If one takes into the equivalence of spaces given by Lemma 4.2, then the relation (4.4) for the operator P results in

$$\begin{aligned}(P_0, \varphi_2)_{H_{\bar{h}_2}} \equiv \iint_\Omega & \left[\nabla_k^2 \left[\bar{h}_2\varphi_2\right] \left(\bar{h}_1\bar{g}_1 + u^*\right) - a_2 K\left(\bar{h}_2\bar{g}_2, \bar{h}_2\varphi_2\right)\right. \\ & - Q_3\left(\bar{h}_1\bar{g}_1, \bar{h}_1\bar{g}_1, \bar{h}_2\varphi_2\right) - Q_3\left(\bar{h}_1\bar{g}_1, h_1^*\bar{g}_1^*, \bar{h}_2\varphi_2\right) - Q_3\left(\bar{h}_1\bar{g}_1, \bar{h}_1^* g_1^*, \bar{h}_2\varphi_2\right) \\ & - Q_3\left(h_1^*\bar{g}_1^*, \bar{h}_1\bar{g}_1, \bar{h}_2\varphi_2\right) - Q_3\left(h_1^*\bar{g}_1^*, h_1^*\bar{g}_1^*, \bar{h}_2\varphi_2\right) - Q_3\left(h_1^*\bar{g}_1^*, \bar{h}_1^* g_1^*, \bar{h}_2\varphi_2\right) \\ & \left. - Q_3\left(\bar{h}_1^* g_1^*, h_1^*\bar{g}_1^*, \bar{h}_2\varphi_2\right) - Q_3\left(h_1^*\bar{g}_1^*, h_1^*\bar{g}_1^*, \bar{h}_2\varphi_2\right) - Q_3\left(h_1^*\bar{g}_1^*, \bar{h}_1\bar{g}_1, \bar{h}_2\varphi_2\right)\right] dx_1 dx_2;\end{aligned}$$

$$\begin{aligned}(P_1, \varphi_2)_{H_{\bar{h}_2}} \equiv \iint_\Omega & \left[\nabla_k^2 \left[\bar{h}_2\varphi_2\right] \bar{h}_1 g_1 - Q_3\left(\bar{h}_1\bar{g}_1, \bar{h}_1 g_1, \bar{h}_2\varphi_2\right)\right. \\ & - Q_3\left(\bar{h}_1 g_1, \bar{h}_1\bar{g}_1, \bar{h}_2\varphi_2\right) - Q_3\left(\bar{h}_1 g_1, h_1^*\bar{g}_1^*, \bar{h}_2\varphi_2\right) - Q_3\left(h_1^*\bar{g}_1^*, \bar{h}_1 g_1, \bar{h}_2\varphi_2\right) \\ & \left. - Q_3\left(\bar{h}_1 g_1, \bar{h}_1^* g_1^*, \bar{h}_2\varphi_2\right)\right] dx_1 dx_2;\end{aligned} \tag{4.14}$$

$$(P_2, \varphi_2)_{H_{\bar{h}_2}} \equiv -\iint_\Omega Q_3\left(\bar{h}_1 g_1, \bar{h}_1 g_1, \bar{h}_2\varphi_2\right) dx_1 dx_2,$$

where the Riesz theorem implies the existence of $P_0, P_1, P_2 \in H_{\bar{h}_2}$. Here, the operator Q_3, introduced for the sake of convenience, is defined as

$$Q_3\left(\alpha, \beta, \gamma\right) = \sum_{p=1}^{P} c_p \frac{\partial^{s_p}\alpha}{\partial x_1^{s_p^{(1)}} \partial x_2^{s_p^{(2)}}} \frac{\partial^{q_p}\beta}{\partial x_1^{q_p^{(1)}} \partial x_2^{q_p^{(2)}}} \frac{\partial^{t_p}\gamma}{\partial x_1^{t_p^{(1)}} \partial x_2^{t_p^{(2)}}}$$

(it coincides with $Q_2\left(\alpha, \beta\right)$ for $\alpha = \beta$).

Substitution of (4.14) to (4.4) leads to the following expression to determine G_s:

$$\begin{aligned}(G_0, \varphi_2)_{H_{\bar{h}_1}} \equiv \iint_\Omega & \left[\nabla_k^2 \left[\bar{h}_2\bar{g}_2\right] \bar{h}_1\varphi_1 - a_1 K\left(\bar{h}_1\bar{g}_1, \bar{h}_1\varphi_1\right) - a_1 K\left(h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, \bar{h}_1\varphi_1\right)\right. \\ & - Q_1\left(\bar{h}_1\bar{g}_1, \bar{h}_2\bar{g}_2, \bar{h}_1\varphi_1\right) - Q_1\left(\bar{h}_1\bar{g}_1, \bar{h}_2 P_0, \bar{h}_1\varphi_1\right) - Q_1\left(h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, \bar{h}_2\bar{g}_2, \bar{h}_1\varphi_1\right) \\ & \left. - Q_1\left(h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, \bar{h}_2 P_0, \bar{h}_1\varphi_1\right)\right] dx_1 dx_2;\end{aligned}$$

$$(G_1,\varphi_2)_{H_{\bar h_1}} \equiv -\iint_\Omega [Q_1\left(\bar h_1\bar g_1, \bar h_2 P_1, \bar h_1\varphi_1\right) + Q_1\left(\bar h_1 g_1, \bar h_2\bar g_2, \bar h_1\varphi_1\right) \tag{4.15}$$
$$+ Q_1\left(\bar h_1 g_1, \bar h_2 P_0, \bar h_1\varphi_1\right) + Q_1\left(h_1^*\bar g_1^* + \bar h_1^* g_1^*, \bar h_2 P_1, \bar h_1\varphi_1\right)]dx_1dx_2;$$

$$(G_2,\varphi_2)_{H_{\bar h_1}} \equiv -\iint_\Omega [Q_1\left(\bar h_1\bar g_1, \bar h_2 P_2, \bar h_1\varphi_1\right) + Q_1\left(\bar h_1 g_1, \bar h_2 P_1, \bar h_1\varphi_1\right)$$
$$+ Q_1\left(h_1^*\bar g_1^* + \bar h_1^* g_1^*, \bar h_2 P_2, \bar h_1\varphi_1\right)]dx_1dx_2;$$

$$(G_3,\varphi_2)_{H_{\bar h_1}} \equiv -\iint_\Omega Q_1\left(\bar h_1 g_1, \bar h_2 P_2, \bar h_1\varphi_1\right) dx_1dx_2.$$

□

Similar representations take place for the operators N and H.

Remark 4.1. One should emphasize that the operators P_s (N_s),$s = 0,1,2$, and G_s (H_s), $s = 0,1,2,3$, are determined with the employment of the spaces $H_{\bar h_1}$, $H_{\bar h_2}(H_{\bar g_1}, H_{\bar g_2})$, (not $H_{\bar h_1^{(i)}}, H_{\bar h_2^{(i)}}$ $H_{\bar g_1^{(i)}}$,$H_{\bar g_2^{(i)}}$), that is possible due to the equivalence of the respective spaces provided by Lemma 4.2.

Theorem 4.3. *Each operator P,P_s $(N$,$N_s)$, $s = 1,2$, maps $H_{\bar h_1}$ into $H_{\bar h_2}$ ($H_{\bar g_1}$ into $H_{\bar g_2}$) strongly continuously, if the sequences $\left\{h_1^{(i)}\right\}$, $\left\{g_1^{(i)}\right\}$ are bounded in $W_2^{(m)}\left(\left[x_1^{\min}; x_1^{\max}\right]\right)$, $W_2^{(m)}\left(\left[x_2^{\min}; x_2^{\max}\right]\right)$, respectively.*

Proof. The sequence $\left\{g_1^{(i)}\right\}$ ($\left\{h_1^{(i)}\right\}$) is bounded infinite set of the Hilbert space $W_2^{(m)}$; therefore it is feebly compact that is one can extract a weakly-converging sequence from its infinite part. A weak limit of the sequence is $g_1(h_1)$. In virtue of Lemma 4.2, the operator P_s (N_s) maps the space $H_{\bar h_1}$ $(H_{\bar g_1})$ into the space $H_{\bar h_2}(H_{\bar g_2})$.

To prove that the operators P_s are strongly continuous it is necessary to prove that if the sequence $g_1^{(i)}$ converges weakly to g_1 in $H_{\bar h_1}$ ($g_1^{(i)} \to g_1$), then the sequence $P_s g_1^{(i)}$ converges strongly in $H_{\bar h_2}$ ($\left\|P_s g_1^{(i)} - P_s g_1\right\|_{H_{\bar h_2}} \to 0$). One should stress that the theorem conditions implies the weak convergence both $g_1^{(i)} \to g_1$ and $h_1^{(i)} \to h_1$.

Firstly, one should study the action of the operator P_1 from $H_{\bar h_1}$ into $H_{\bar h_2}$. The addenda of the difference $\left|\left(P_1 g_1^{(i)} - P_1 g_1, \varphi_2\right)_{H_{\bar h_2}}\right|$ (accounting for (4.14)) can be presented as

1.

$$\left|\iint_\Omega \nabla_k^2 \left[\bar h_2^{(i)}\varphi_2\right] \bar h_1^{(i)} g_1^{(i)} dx_1dx_2 - \iint_\Omega \nabla_k^2 \left[\bar h_2\varphi_2\right] \bar h_1 g_1 dx_1dx_2\right|;$$

2.

$$\left|\iint_\Omega Q_3\left(\bar h_1\bar g_1, \bar h_1 g_1, \bar h_2\varphi_2\right) dx_1dx_2 - \iint_\Omega Q_3\left(\bar h_1^{(i)}\bar g_1^{(i)}, \bar h_1^{(i)} g_1^{(i)}, \bar h_2^{(i)}\varphi_2\right) dx_1dx_2\right|;$$

3.

$$\left|\iint_\Omega Q_3\left(\bar h_1 g_1, \bar h_1\bar g_1, \bar h_2\varphi_2\right) dx_1dx_2 - \iint_\Omega Q_3\left(\bar h_1^{(i)} g_1^{(i)}, \bar h_1^{(i)}\bar g_1^{(i)}, \bar h_2^{(i)}\varphi_2\right) dx_1dx_2\right|; \tag{4.16}$$

4.

$$\left|\iint_\Omega Q_3\left(\bar h_1 g_1, h_1^*\bar g_1^*, \bar h_2\varphi_2\right) dx_1dx_2 - \iint_\Omega Q_3\left(\bar h_1^{(i)} g_1^{(i)}, h_1^*\bar g_1^*, \bar h_2^{(i)}\varphi_2\right) dx_1dx_2\right|;$$

5.

$$\left|\iint_\Omega Q_3\left(h_1^*\bar g_1^*, \bar h_1 g_1, \bar h_2\varphi_2\right) dx_1dx_2 - \iint_\Omega Q_3\left(h_1^*\bar g_1^*, \bar h_1^{(i)} g_1^{(i)}, \bar h_2^{(i)}\varphi_2\right) dx_1dx_2\right|;$$

6.

$$\left|\iint_\Omega Q_3\left(\bar h_1 g_1, \bar h_1^* g_1^*, \bar h_2\varphi_2\right) dx_1dx_2 - \iint_\Omega Q_3\left(\bar h_1^{(i)} g_1^{(i)}, \bar h_1^* g_1^*, \bar h_2^{(i)}\varphi_2\right) dx_1dx_2\right|.$$

For the first term, the following holds true:

$$\begin{aligned}
&|\iint_\Omega \nabla_k^2 \left[\bar h_2^{(i)}\varphi_2\right] \bar h_1^{(i)} g_1^{(i)} dx_1dx_2 - \iint_\Omega \nabla_k^2 \left[\bar h_2\varphi_2\right] \bar h_1 g_1 dx_1dx_2| = \\
&|\iint_\Omega \nabla_k^2 \left[\left(\bar h_2 - \bar h_2^{(i)}\right)\varphi_2\right] \bar h_1 g_1 dx_1dx_2 + \iint_\Omega \nabla_k^2 \left[\bar h_2^{(i)}\varphi_2\right] \left[\bar h_1^{(i)} g_1^{(i)} - \bar h_1 g_1\right] dx_1dx_2| \\
&\leqslant \left\|\nabla_k^2 \left[\left(\bar h_2 - \bar h_2^{(i)}\right)\varphi_2\right]\right\|_{L_2(\Omega)} \left\|\bar h_1 g_1\right\|_{L_2(\Omega)} \\
&+ \left\|\nabla_k^2 \left[\bar h_2^{(i)}\varphi_2\right]\right\|_{L_2(\Omega)} \left[\left\|\bar h_1^{(i)}\right\|_{L_4(\Omega)} \left\|g_1^{(i)} - g_1\right\|_{L_4(\Omega)} + \left\|g_1\right\|_{L_4(\Omega)} \left\|\bar h_1^{(i)} - \bar h\right\|_{L_4(\Omega)}\right]
\end{aligned}$$

(owing to the embedding theorems 2.1 and 2.5)

$$\begin{aligned}
&\leqslant M \left\|\bar h_1\right\|_{L_4} \left\|g_1\right\|_{L_4} \left[\left\|\nabla_k^2 \left[\left(\bar h_2 - \bar h_2^{(i)}\right)\right]\right\|_{L_8} + \left\|\bar h_2 - \bar h_2^{(i)}\right\|_{L_8}\right] \left\|\varphi_2\right\|_{H_{\bar h_2}} \\
&+ \left[\left\|\bar h_1^{(i)}\right\|_{W_2^{(m)}} \left\|g_1^{(i)} - g_1\right\|_{L_4} + \left\|g_1\right\|_{W_2^{(m)}} \left\|\bar h_1^{(i)} - \bar h\right\|_{L_4}\right] \left\|\varphi_2\right\|_{H_{\bar h_2}}.
\end{aligned} \tag{4.17}$$

This is grounded on the inequalities following from the Sobolev embedding theorem ($L_8(\Omega) \to W_2^{(m)}(\Omega)$), from the embedding theorems 2.1 and 2.5, and from the Lemma 4.1 as well:

$$\left\{\left\|\nabla_k^2\varphi_2\right\|_{L_8(\Omega)}, \|\varphi_2\|_{L_8(\Omega)}\right\} \leqslant M\|\varphi_2\|_{W_2^{(m)}(\Omega)} = M\|\varphi_2\|_{W_2^{(m)}\left[x_2^{\min},x_2^{\max}\right]} \leqslant M\|\varphi_2\|_{H_{\bar h_2}\left[x_2^{\min},x_2^{\max}\right]}, \tag{4.18}$$

$$\left\|\nabla_k^2\left[\bar h_2^{(i)}\varphi_2\right]\right\|_{L_8(\Omega)} \leqslant M\left\|\bar h_2^{(i)}\varphi_2\right\|_{W_2^{(m)}(\Omega)} \leqslant M\left\|\bar h_2^{(i)}\varphi_2\right\|_{H(\Omega)} = M\|\varphi_2\|_{H_{\bar h_2}\left[x_2^{\min},x_2^{\max}\right]} \leqslant M\|\varphi_2\|_{H_{\bar h_2^{(i)}}\left[x_2^{\min},x_2^{\max}\right]}. \tag{4.19}$$

One should lay emphasis on the fact that, due to strong continuity of the embedding operators proved by theorem 2.1 and 2.5, coefficient for $\|\varphi_2\|_{H_{\bar h_2}}$ in the right-hand term of (4.17) vanishes as $i \to \infty$.

For other terms of (4.16) a structure of $Q_3(\alpha,\beta,\gamma)$ substantiate the following estimates

$$|Q_3(\alpha_1,\beta_1,\gamma_1) - Q_3(\alpha_2,\beta_2,\gamma_2)| \leqslant \sum_{p=0}^{P} c_p\left|Q_3^{(p)}(\alpha_1,\beta_1,\gamma_1) - Q_3^{(p)}(\alpha_2,\beta_2,\gamma_2)\right|$$
$$= \sum_{p=0}^{P} c_p\left[\left|Q_3^{(p)}(\alpha_1-\alpha_2,\beta_1,\gamma_1)\right| + \left|Q_3^{(p)}(\alpha_2,\beta_1-\beta_2,\gamma_2)\right|\left|Q_3^{(p)}(\alpha_2,\beta_1,\gamma_1-\gamma_2)\right|\right],$$

where

$$Q_3^{(p)}(\alpha,\beta,\gamma) = \frac{\partial^{s_p}\alpha}{\partial x_1^{s_p^{(1)}}\partial x_2^{s_p^{(2)}}}\frac{\partial^{q_p}\beta}{\partial x_1^{q_p^{(1)}}\partial x_2^{q_p^{(2)}}}\frac{\partial^{t_p}\gamma}{\partial x_1^{t_p^{(1)}}\partial x_2^{t_p^{(2)}}}.$$

Consequently

$$|Q_3(\alpha_1,\beta_1,\gamma_1) - Q_3(\alpha_2,\beta_2,\gamma_2)| \leqslant \sum_{p=0}^{P} c_p\Big[\left|Q_3^{(p)}(\alpha_1-\alpha_2,\beta_1,\gamma_1)\right|$$
$$+ \left|Q_3^{(p)}(\alpha_2,\beta_1-\beta_2,\gamma_2)\right| + \left|Q_3^{(p)}(\alpha_2,\beta_1,\gamma_1-\gamma_2)\right|$$
$$\leqslant \sum_{p=0}^{P} c_p\Big[\left\|\frac{\partial^{s_p}[\alpha_1-\alpha_2]}{\partial x_1^{s_p^{(1)}}\partial x_2^{s_p^{(2)}}}\right\|_{L_4(\Omega)}\left\|\frac{\partial^{q_p}\beta_1}{\partial x_1^{q_p^{(1)}}\partial x_2^{q_p^{(2)}}}\right\|_{L_4(\Omega)}\left\|\frac{\partial^{t_p}\gamma_1}{\partial x_1^{t_p^{(1)}}\partial x_2^{t_p^{(2)}}}\right\|_{L_2(\Omega)}$$
$$+ \left\|\frac{\partial^{s_p}\alpha_2}{\partial x_1^{s_p^{(1)}}\partial x_2^{s_p^{(2)}}}\right\|_{L_4(\Omega)}\left\|\frac{\partial^{q_p}[\beta_1-\beta_2]}{\partial x_1^{q_p^{(1)}}\partial x_2^{q_p^{(2)}}}\right\|_{L_4(\Omega)}\left\|\frac{\partial^{t_p}\gamma_1}{\partial x_1^{t_p^{(1)}}\partial x_2^{t_p^{(2)}}}\right\|_{L_2(\Omega)}$$
$$+ \left\|\frac{\partial^{s_p}\alpha_2}{\partial x_1^{s_p^{(1)}}\partial x_2^{s_p^{(2)}}}\right\|_{L_4(\Omega)}\left\|\frac{\partial^{q_p}\beta_1}{\partial x_1^{q_p^{(1)}}\partial x_2^{q_p^{(2)}}}\right\|_{L_4(\Omega)}\left\|\frac{\partial^{t_p}[\gamma_1-\gamma_2]}{\partial x_1^{t_p^{(1)}}\partial x_2^{t_p^{(2)}}}\right\|_{L_2(\Omega)}\Big]$$

For all terms 2-6 of (4.16) the variable γ_1 (γ_2) looks like $\bar{h}_2\varphi_2(\bar{h}_2^{(i)}\varphi_2)$, thereby the co-products $\left\| \frac{\partial^{t_p}\gamma_1}{\partial x_1^{t_p^{(1)}} \partial x_2^{t_p^{(2)}}} \right\|_{L_2(\Omega)}$ $\left(\left\| \frac{\partial^{t_p}[\gamma_1-\gamma_2]}{\partial x_1^{t_p^{(1)}} \partial x_2^{t_p^{(2)}}} \right\|_{L_2(\Omega)} \right)$ can be bounded by $\|\varphi_2\|_{H_{\bar{h}_2}}$ $\left(\left\| \frac{\partial^{t_p^{(1)}}\left[\bar{h}_2-\bar{h}_2^{(i)}\right]}{\partial x_1^{t_p^{(1)}}} \right\|_{L_4(\Omega)} \|\varphi_2\|_{H_{\bar{h}_2}} \right)$ using a sequence of inequalities of a kind of (4.19), (4.18). The variables β_1 look like $\beta_1 = \beta_1^{x_1}(x_1)\,\beta_1^{x_2}(x_2)$, and thus can be estimated as $\left\| \frac{\partial^{q_p}\beta_1}{\partial x_1^{q_p^{(1)}} \partial x_2^{q_p^{(2)}}} \right\|_{L_4(\Omega)} \leqslant M \|\beta_1^{x_1}\|_{W_2^{(m)}[x_1^{\min};x_1^{\max}]} \|\beta_2^{x_2}\|_{W_2^{(m)}[x_2^{\min};x_2^{\max}]}$ with the employment of the embedding theorem. Finally, the co-products $\left\| \frac{\partial^{s_p}[\alpha_1-\alpha_2]}{\partial x_1^{s_p^{(1)}} \partial x_2^{s_p^{(2)}}} \right\|_{L_4(\Omega)}$ are equal to zero (if $\alpha_1=\alpha_2$)or can be presented as $\alpha_{x_1}\alpha_{x_2} - \alpha_{x_1}^{(i)}\alpha_{x_2}^{(i)}$, and the following

$$\left\| \frac{\partial^{s_p}\left[\alpha_{x_1}\alpha_{x_2} - \alpha_{x_1}^{(i)}\alpha_{x_2}^{(i)}\right]}{\partial x_1^{s_p^{(1)}} \partial x_2^{s_p^{(2)}}} \right\|_{L_4(\Omega)} \leqslant \|\alpha_{x_1}\|_{W_2^{(m)}(\Omega)} \left\| \frac{\partial^{s_p}\left[\alpha_{x_2} - \alpha_{x_2}^{(i)}\right]}{\partial x_1^{s_p^{(1)}} \partial x_2^{s_p^{(2)}}} \right\|_{L_8(\Omega)} + \|\alpha_{x_2}\|_{W_2^{(m)}(\Omega)} \left\| \frac{\partial^{s_p}\left[\alpha_{x_1} - \alpha_{x_1}^{(i)}\right]}{\partial x_1^{s_p^{(1)}} \partial x_2^{s_p^{(2)}}} \right\|_{L_8(\Omega)}. \tag{4.20}$$

is valid for them.

This leads to the following estimates for each term 2-6 of (4.16):

$$\sum_{p=0}^{P} c_p\Bigg[\left[\|\alpha_{x_1}\|_{W_2^{(m)}} \left\| \frac{\partial^{s_p^{(2)}}\left[\alpha_{x_2} - \alpha_{x_2}^{(i)}\right]}{\partial x_2^{s_p^{(2)}}} \right\|_{L_8} + \|\alpha_{x_2}\|_{W_2^{(m)}} \left\| \frac{\partial^{s_p^{(1)}}\left[\alpha_{x_1} - \alpha_{x_1}^{(i)}\right]}{\partial x_1^{s_p^{(1)}}} \right\|_{L_8} \right] \tag{4.21}$$
$$\|\beta_{x_1}\|_{W_2^{(m)}} \|\beta_{x_2}\|_{W_2^{(m)}}$$
$$+ \left[\|\beta_{x_1}\|_{W_2^{(m)}} \left\| \frac{\partial^{q_p^{(2)}}\left[\beta_{x_2} - \beta_{x_2}^{(i)}\right]}{\partial x_2^{q_p^{(2)}}} \right\|_{L_8} + \|\beta_{x_2}\|_{W_2^{(m)}} \left\| \frac{\partial^{q_p^{(1)}}\left[\beta_{x_1} - \beta_{x_1}^{(i)}\right]}{\partial x_1^{q_p^{(1)}}} \right\|_{L_8} \right]$$
$$+ \|\alpha_{x_1}\|_{W_2^{(m)}} \|\alpha_{x_2}\|_{W_2^{(m)}} \|\beta_{x_1}\|_{W_2^{(m)}} \|\beta_{x_2}\|_{W_2^{(m)}} \left\| \frac{\partial^{t_p^{(1)}}\left[\gamma_{x_1} - \gamma_{x_1}^{(i)}\right]}{\partial x_1^{t_p^{(1)}}} \right\|_{L_4} \Bigg] \|\varphi_2\|_{H_{\bar{h}_2}}$$

To summarize, as for the first term, for each term 2-6, coefficient for $\|\varphi_2\|_{H_{\bar{h}_2}} \to 0$, if $g_1^{(i)} \to g_1$ and $h_1^{(i)} \to h_1$ in $W_2^{(m)}$ with $i \to \infty$. Therefore, the aggregated coefficients for $\|\varphi_2\|_{H_{\bar{h}_2}}$ and thereby $\left\|P_1 g_1^{(i)} - P_1 g_1\right\|_{H_{\bar{h}_2}}$ (the infimum of such coefficients), vanishes as $i \to \infty$. This proves strong continuity of the operator P_1.

For the operator P_2, mapping $H_{\bar{h}_1}$ into $H_{\bar{h}_2}$, (4.14) results in

$$\left(P_2 g_1^{(i)} - P_2 g_1, \varphi_2\right)_{H_{\bar{h}_2}} \leqslant \sum_{p=0}^{P} c_p [\|\bar{h}_1\|_{W_2^{(m)}} \|g_1\|_{W_2^{(m)}} \left\| \frac{\partial^{q_p}\left[\bar{h}_1 g_1\right]}{\partial x_1^{q_p^{(1)}} \partial x_2^{q_p^{(2)}}} - \frac{\partial^{q_p}\left[\bar{h}_1^{(i)} g_1^{(i)}\right]}{\partial x_1^{q_p^{(1)}} \partial x_2^{q_p^{(2)}}} \right\|_{L_4} + \left\|\bar{h}_1^{(i)}\right\|_{W_2^{(m)}} \left\|g_1^{(i)}\right\|_{W_2^{(m)}} \left\| \frac{\partial^{p_p}\left[\bar{h}_1 g_1\right]}{\partial x_1^{p_p^{(1)}} \partial x_2^{p_p^{(2)}}} - \frac{\partial^{p_p}\left[\bar{h}_1^{(i)} g_1^{(i)}\right]}{\partial x_1^{p_p^{(1)}} \partial x_2^{p_p^{(2)}}} \right\|_{L_4} \Bigg] \|\varphi_2\|_{H_{\bar{h}_2}(\Omega)}$$

Inequality (4.20) and the theorem 2.1 ensures that the coefficient for $\|\varphi_2\|_{H_{\bar{h}_2}}$ vanishes for that case likewise, and this completes the proof for P_s. Similar proof allows establishing strong continuity of operators N_s. □

Remark 4.2. The previous theorem proves strong convergence $P_1 g_1^{(i)} \Rightarrow P_1 g_1$ in $H_{\bar{h}_2}$ and, consequently, due to the lemma 4.2, in $W_2^{(m)}$.

Theorem 4.4. *If the conditions of the theorem 4.3 hold true, then each operator* G,H,G_s,$H_s s = 1, 2, 3$, *maps strongly continuously* $H_{\bar{h}_1}$, $H_{\bar{g}_1}$ *into itself, respectively.*

Proof. With results similar to those of the previous theorem, the operators G_s are proved to map $H_{\bar{h}_1}$ into itself.

If $g_1^{(i)}$ converges weakly to g_1 in $H_{\bar{h}_1}$ ($g_1^{(i)} \rightarrow g_1$); then it is required to prove that the sequence $G_s g_1^{(i)}$ converges strongly to the limit indicated ($\left\|G_s g_1^{(i)} - g_1\right\|_{H_{\bar{h}_1}} \rightarrow 0$). As for the previous theorem, one considers only the numbers corresponding to the weakly-converged (in $H_{\bar{h}_1}$) sequence; the theorem implies that the sequences $\left\{h_1^{(i)}\right\}$, $\left\{g_1^{(i)}\right\}$ are bounded in $W_2^{(m)}\left(\left[x_1^{\min}; x_1^{\max}\right]\right)$, $W_2^{(m)}\left(\left[x_2^{\min}; x_2^{\max}\right]\right)$.

The relation (4.15) leads to the following terms of the difference $\left|\left(G_1 g_1^{(i)} - G_1 g_1, \varphi_2\right)_{H_{\bar{h}_1}}\right|$:

1.

$$| \iint_\Omega Q_1\left(\bar{h}_1 \bar{g}_1, \bar{h}_2 P_1 g_1, \bar{h}_1 \varphi_1\right) dx_1 dx_2 - \iint_\Omega Q_1\left(\bar{h}_1^{(i)} \bar{g}_1^{(i)}, \bar{h}_2^{(i)} P_1 g_1^{(i)}, \bar{h}_1^{(i)} \varphi_1\right) dx_1 dx_2|;$$

2.

$$\left|\iint_\Omega Q_1\left(\bar{h}_1 g_1, \bar{h}_2 \bar{g}_2, \bar{h}_1 \varphi_1\right) dx_1 dx_2 - \iint_\Omega Q_1\left(\bar{h}_1^{(i)} g_1^{(i)}, \bar{h}_2^{(i)} g_2^{(i)}, \bar{h}_1^{(i)} \varphi_1\right) dx_1 dx_2\right|;$$

3.

$$\left|\iint_\Omega Q_1\left(\bar{h}_1 g_1, \bar{h}_2 P_0, \bar{h}_1\varphi_1\right) dx_1 dx_2 - \iint_\Omega Q_1\left(\bar{h}_1^{(i)} g_1^{(i)}, \bar{h}_2^{(i)} P_0, \bar{h}_1^{(i)}\varphi_1\right) dx_1 dx_2\right|;$$

4.

$$| \iint_\Omega Q_1\left(h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, \bar{h}_2 P_1, \bar{h}_1\varphi_1\right) dx_1 dx_2 - \iint_\Omega Q_1\left(h_1^*\bar{g}_1^* + \bar{h}_1^* g_1^*, \bar{h}_2^{(i)} P_1, \bar{h}_1^{(i)}\varphi_1\right) dx_1 dx_2|.$$

For terms 2 and 3, the proof that the coefficients for $\|\varphi_1\|_{H_{\bar{h}_1}}$ vanishes (if the theorem conditions are true) is carried out along similar lines with that for terms defining the operator P_1. For terms 1 and 4 the proof is performed analogously but the relation (4.21) comprises the term $\left\|\frac{\partial^{q_p^{(1)}}\left[P_1 g_1 - P_1 g_1^{(i)}\right]}{\partial x_1^{q_p^{(1)}}}\right\|_{L_8}$; to prove that such term vanishes one should apply the embedding theorem 2.1 and the previous theorem that examines convergence of the sequence $P_1 g_1^{(i)}$ in $W_2^{(m)}$ (see Remark 4.2).

The above considerations allow bounding $\left\|G_1 g_1^{(i)} - G_1 g_1\right\|_{H_{\bar{h}_1}}$ by a vanishing expression and thus proving strong continuity of the operator G_1.That of the operators G_2,G_3, H_s is proved analogously. □

Strong continuity of both operators G and H leads to the following theorem:

Theorem 4.5. *If the sequences of approximations of the iterative generalized Kantorovich method* $\left\{g_1^{(i)}\right\}$, $\left\{h_1^{(i)}\right\}$ *belong to bounded regions of spaces* H_{h_1}, H_{g_1}, *respectively. Then the sequence*

$$\left\{u_j^{(i)}(x_1, x_2) = h_{\mathrm{j}}^{(\mathrm{i})}(x_1)\,\bar{g}_j^{(i)}(x_2) + \bar{h}_{\mathrm{j}}^{(\mathrm{i})}(x_1)\,g_j^{(i)}(x_2)\right\}$$

converges strongly in H.

Proof. According to theorem conditions, the sequences $\left\{g_1^{(i)}\right\}$, $\left\{h_1^{(i)}\right\}$ are bounded infinite sets of Hilbert spaces H_{h_1}, H_{g_1}; therefore they are feebly compact. As usual one confines oneself to the numbers belonging to the subsequences in question with g_1 and h_1 as their (weak) limits.

The sequences $\left\{g_1^{(i)}\right\}$, $\left\{h_1^{(i)}\right\}$ are the result of action of the operators G and H on the weakly-converging sequences $\left\{g_1^{(i)}\right\}$ and $\left\{h_1^{(i)}\right\}$. Since the operators are strongly continuous in virtue of the theorem 4.4, the sequences $\left\{g_1^{(i)}\right\}$, $\left\{h_1^{(i)}\right\}$ converge strongly in these spaces. Theorem 4.3 proves the strong convergence of the sequences $\left\{g_2^{(i)}\right\}$, $\left\{h_2^{(i)}\right\}$ likewise. The inequality $\left\|h_j^{(i)}g_j^{(i)} - h_jg_j\right\|_H \leqslant m\left[\left\|g_j^{(i)} - g_j\right\|_{H_{g_1}} + \left\|h_j^{(i)} - h_j\right\|_{H_{g_1}}\right]$ completes the proof. □

In actual practice, the criterion to complete the iteration process is that the norms of differences of solutions are small for several (three to four) consecutive iterations that allow checking the assumption that the above-mentioned sequences are bounded. The method demonstrated good practical convergence. It took $6-8$ iterations to converge for maximum relative accuracy 10^{-4}.

5. Conclusions

1. The correctness of definition is proved for the non-linear elliptic of von Karman-type equations.

2. It is ascertained that the iterative generalized Kantorovich method is applicable to solve the non-linear elliptic of von Karman-type equations.

3. It is proved that the approximate sequence generated by the method in question converges strongly in specific energy space H.

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