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Проблеми математичного моделювання
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ON EXISTENCE OF OPTIMAL CONTROLS IN COEFFICIENTS FOR ILL-POSED NONLINEAR ELLIPTIC DIRICHLET BOUNDARY VALUE PROBLEMS WITH ANISOTROPIC P-LAPLACIAN

O. P. Kupenko

Department of System Analysis and Control, National Mining University, Yavornitskii av., 19, 49005 Dnipropetrovsk, Ukraine; Institute for Applied System Analysis National Academy of Sciences and Ministry of Education and Science of Ukraine, Kyiv, e-mail: olga.kupenko@bk.ru.

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We study a Dirichlet optimal control problem for a nonlinear elliptic anisotropic p-Laplace equation with control and state constraints. The matrix-valued coefficients $A_{sym} \in L^\infty(\Omega; \mathbb{S}^N_{sym})$ we take as controls and in the linear part of differential operator we consider coefficients to be unbounded skew-symmetric matrix $A_{skew} \in L^q(\Omega; \mathbb{S}^N_{skew})$. We show that, in spite of unboundedness of the non-linear differential operator, the considered Dirichlet problem admits at least one weak solution and the corresponding OCP is well-possed and solvable.

Keywords: optimal control, anisotropic elliptic equation, control in coefficients, existence result.

1. Introduction

In this paper we deal with the following optimal control problem (OCP) for nonlinear elliptic equation with unbounded coefficients

$$\text{Minimize } I(A, y) = \|y - y_d\|^p_{L^p(\Omega)} + \varepsilon \int_\Omega |A^{\frac{1}{2}}_{sym} \nabla y|^p_{\mathbb{R}^N}\, dx \tag{1.1}$$

subject to the constraints

$$-\mathrm{div}\left(|(A_{sym}\nabla y, \nabla y)|^{\frac{p-2}{2}} A_{sym}\nabla y\right) - \mathrm{div}\left(A_{skew}\nabla y\right) = -\mathrm{div}\, f, \tag{1.2}$$

$$A_{sym} \in \mathfrak{A}_{ad},\ y \in W^{1,p}_0(\Omega) \tag{1.3}$$

where p satisfies $2 < p < +\infty$, $\varepsilon > 0$ is a small fixed parameter, the symmetric matrix of anisotropy $A_{sym} \in L^\infty(\Omega; \mathbb{R}^{N\times N}) \cap BV(\Omega; \mathbb{R}^{N\times N})$ is taken as a control, the skew-symmetric matrix $A_{skew} \in L^q(\Omega; \mathbb{R}^{N\times N})$ is a given matrix of coefficients, $y_d \in L^p(\Omega)$ and $f \in L^q(\Omega; \mathbb{R}^N)$ are given distributions. Here $\mathfrak{A}_{ad}$ is a class of admissible controls, which is a nonempty subset of $L^\infty(\Omega; \mathbb{R}^{N\times N}) \cap BV(\Omega; \mathbb{R}^{N\times N})$.

Thereby, for a "typical" symmetric control we deal with the Dirichlet boundary value problem for non-degenerate anisotropic elliptic equation (see [2]). On the other hand, the unboundedness of skew-symmetric matrix can lead to the existence of elements $y \in W_0^p(\Omega)$ such that $y \notin L^\infty(\Omega)$,

$$\int_\Omega \left(\nabla\varphi_n, A_{skew}\nabla\varphi_n\right)_{\mathbb{R}^N} dx = 0 \; \forall\, n \in \mathbb{N}, \text{ and } \lim_{n\to\infty}\int_\Omega \left(\nabla\varphi_n, A_{skew}\nabla y\right)_{\mathbb{R}^N} dx \neq 0$$

where $C_0^\infty(\Omega) \ni \varphi_n \to y$ strongly in $W_0^p(\Omega)$ [6, 7, 15]. As a result, the existence, uniqueness, and variational properties of the weak solution to elliptic equation (1.2) usually are drastically different from the corresponding properties of solutions to the elliptic equations with coercive L^∞-matrices in coefficients (we refer to [3, 10–13] for the details and other results in this field).

We will prove that considered elliptic Dirichlet problem admits at least one weak solution in the sense of Minty and the corresponding optimal control problem (1.1) has a nonempty set of weak solutions (see Theorem 3.1).

2. Notation and Preliminaries

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$. Let χ_E be the characteristic function of a subset $E \subset \Omega$, i.e. $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$.

Let $\mathbb{M}^N$ be the set of all $N \times N$ real matrices. We denote by $\mathbb{S}^N_{skew}$ the set of all skew-symmetric matrices $C = [c_{ij}]^N_{i,j=1}$. Thus, if $C \in \mathbb{S}^N_{skew}$ then $c_{ij} = -c_{ji}$ and, hence, $c_{ii} = 0$. Let $\mathbb{S}^N_{sym}$ be the set of all $N \times N$ symmetric matrices. By matrix norm in $\mathbb{M}^N$ (and for functions with values in $\mathbb{S}^N_{sym}$ and $\mathbb{S}^N_{skew}$ as well) we mean a sub-multiplicative norm $\|A\| := \sup\left\{|A\xi|_{\mathbb{R}^N} \;:\; \xi \in \mathbb{R}^N \text{ with } |\xi|_{\mathbb{R}^N} = 1\right\}$. It is worth to note that, in the case of Euclidean norm $|\cdot|_{\mathbb{R}^N}$, the norm $\|A\|$ can be computed as the spectral norm $\|A\| = \sqrt{\lambda_{max}(A^t A)}$, where $\lambda_{max}(A^t A)$ is the largest eigenvalue of the positive-semidefinite matrix $A^t A$.

Let $p > 2$ be given real number and $1 < q < \infty$ be its conjugate such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $L^q\left(\Omega; \mathbb{S}^N_{skew}\right)$ be the normed space of measurable integrable with the power q functions whose values are skew-symmetric matrices.

Let $L^1(\Omega)^{\frac{N(N+1)}{2}} = L^1\left(\Omega; \mathbb{S}^N_{sym}\right)$ be the space of integrable functions whose values are symmetric matrices. By $BV(\Omega; \mathbb{S}^N_{sym})$ we denote the space of all matrices in $L^1(\Omega; \mathbb{S}^N_{sym})$ for which the norm

$$\|C\|_{BV(\Omega;\mathbb{S}^N_{sym})} = \|C\|_{L^1(\Omega;\mathbb{S}^N_{sym})} + \int_\Omega |DC| = \|C\|_{L^1(\Omega;\mathbb{S}^N_{sym})}$$
$$+ \sum_{1\leq i\leq j\leq N} \sup\left\{ \int_\Omega c_{ij} \operatorname{div}\varphi\, dx \;:\; \varphi \in C_0^1(\Omega;\mathbb{R}^N),\; |\varphi(x)| \leq 1 \text{ for } x \in \Omega\right\} \quad (2.1)$$

is finite.

Remark 2.1. We recall that a sequence $\{f_k\}_{k=1}^{\infty}$ converges weakly* to f in $BV(\Omega)$ if and only if two following conditions hold (see [1]): $f_k \to f$ strongly in $L^1(\Omega)$ and $Df_k \overset{*}{\rightharpoonup} Df$ weakly* in the space of Radon measures $\mathcal{M}(\Omega;\mathbb{R}^N)$. Each uniformly bounded set in $BV(\Omega)$ is compact in $L^1(\Omega)$ with respect to the strong topology in this space. Moreover, if $\{f_k\}_{k=1}^{\infty} \subset BV(\Omega)$ converges strongly to some f in $L^1(\Omega)$ and satisfies $\sup_{k\in\mathbb{N}} \int_\Omega |Df_k| < +\infty$, then (see, for instance, [1])

$$\begin{aligned} &(i)\ f \in BV(\Omega) \ \text{ and } \int_\Omega |Df| \le \liminf_{k\to\infty} \int_\Omega |Df_k|; \\ &(ii)\ f_k \overset{*}{\rightharpoonup} f \ \text{ in } \ BV(\Omega). \end{aligned} \tag{2.2}$$

Let ξ_1, ξ_2 be given elements of $L^\infty(\Omega)\cap BV(\Omega)$ satisfying the conditions

$$0 < \alpha \le \xi_1(x) \le \xi_2(x) \text{ a.e. in } \Omega, \tag{2.3}$$

where α is a given positive value. Let $y_d \in L^p(\Omega)$ and $f \in L^q(\Omega;\mathbb{R}^N)$ be given distributions. We define the class of admissible controls $\mathfrak{A}_{ad}$ as follows

$$\mathfrak{A}_{ad} = \left\{ A \in L^\infty(\Omega;\mathbb{S}^N_{sym}) \;\middle|\; \begin{array}{c} \xi_1^2\|\eta\|^2 \le (\eta, A\eta) \le \xi_2^2\|\eta\|^2 \text{ a.e. in } \Omega\ \forall\,\eta\in\mathbb{R}^N, \\ A^{\frac12} \in BV(\Omega;\mathbb{S}^N_{sym}),\ \int_\Omega |DA^{\frac12}| \le \gamma \end{array} \right\}, \tag{2.4}$$

where $\gamma > 0$ is a given constant. In view of estimates

$$\begin{aligned} \|A^{\frac12}(x)\| &\le \xi_2(x) \ \text{ a.e. in } \Omega, \\ \int_\Omega \|A\|^{\frac{p}{2}}\,dx \le \int_\Omega \|A^{\frac12}\|^p, dx &\le \int_\Omega \|A^{\frac12}\|^{p-1}\|A^{\frac12}\|\,dx \\ &\le \|\xi_2\|^{p-1}_{L^\infty(\Omega)} \int_\Omega \|A^{\frac12}\|\,dx, \end{aligned}$$

it is clear that $\mathfrak{A}_{ad}$ is a nonempty convex subset of $L^{\frac{p}{2}}(\Omega;\mathbb{S}^N_{sym})$ with empty topological interior.

Remark 2.2. It is worth to mention, that any sequence of admissible controls $\{A_k\}_{k\in\mathbb{N}} \subset \mathfrak{A}_{ad}$ possesses the following property: sequence $\{A_k^{\frac12}\}_{k\in\mathbb{N}}$ is compact with respect to weak-$*$ convergence in $BV(\Omega;\mathbb{S}^N_{sym})$, i.e. there exists an element $A_0 \in \mathfrak{A}_{ad}$ such that within a subsequence $A_k^{\frac12} \overset{*}{\rightharpoonup} A_0^{\frac12}$ in $BV(\Omega;\mathbb{S}^N)$. Indeed, the above mentioned sequence is weakly-$*$ compact in $L^\infty(\Omega;\mathbb{S}^N_{sym})$, as far as $\|A_k^{\frac12}\|_{L^\infty(\Omega;\mathbb{S}^N_{sym})} \le \|\xi_2\|_{L^\infty(\Omega)}$, and, hence, it is weakly compact in $L^1(\Omega;\mathbb{S}^N_{sym})$ and $\|A_k^{\frac12}\|_{L^1(\Omega;\mathbb{S}^N_{sym})} \le |\Omega|\|\xi_2\|_{L^\infty(\Omega)}$. This fact together with (2.4) gives uniform boundedness $\|A_k^{\frac12}\|_{BV(\Omega;\mathbb{S}^N_{sym})} \le |\Omega|\|\xi_2\|_{L^\infty(\Omega)}+\gamma$ and desirable property immediately follows with respect to Remark 2.1

The optimal control problem we consider in this paper is to minimize the discrepancy between the distribution $y_d \in L^p(\Omega)$ and the solutions $y \in W_0^{1,p}(\Omega)$

of the boundary value problem (1.2) by choosing an appropriate matrix-valued function $A_{sym} \in \mathfrak{A}_{ad}$ as control. Here, by $\Delta_p(A_{sym}, \cdot) : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ we denote the so-called anisotropic p-Laplacian which usually can be defined by the rule (see [2] and references therein)

$$\Delta_p(A_{sym}, y) = -\mathrm{div}\left(|(A_{sym}\nabla y, \nabla y)|^{\frac{p-2}{2}} A_{sym}\nabla y\right),$$

and $A_{skew} \in L^q\left(\Omega; \mathbb{S}^N_{skew}\right)$ is a given matrix, $f \in L^q(\Omega; \mathbb{R}^N)$ is a given distribution.

In view of (2.4) it is easy to see that $\Delta_p(A_{sym}, \cdot) : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ is a bounded operator possessing semicontinuity, coercivity and strict monotonicity properties. Let V be a reflexive separable Banach space , V^* be its dual. We recall, that operator $A : V \to V^*$ is called

- monotone if

$$\langle \mathcal{A}y - \mathcal{A}v, y - v\rangle_{V^*;V} \geq 0, \quad \forall\, y, v \in V; \tag{2.5}$$

- strictly monotone if (2.5) holds and

$$\langle \mathcal{A}y - \mathcal{A}v, y - v\rangle_{V^*;V} = 0 \implies y = v; \tag{2.6}$$

- semicontinuous, if

$$\text{the function } \ \mathbb{R} \ni t \mapsto \langle \mathcal{A}(y + tv), w\rangle_{V^*;V} \ \text{ is continuous for all } \ y, v, w \in V. \tag{2.7}$$

- coercive, if

$$\frac{\langle Ay, y\rangle_{V^*;V}}{\|y\|_V} \to +\infty \text{ as } \|y\|_V \to +\infty. \tag{2.8}$$

Indeed,

$$\begin{aligned}\langle \Delta_p(A_{sym}, \varphi), v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} :&= \int_\Omega |(A_{sym}\nabla\varphi, \nabla\varphi)|^{\frac{p-2}{2}} \left(A_{sym}\nabla\varphi, \nabla v\right)\, dx \\ &= \int_\Omega |A_{sym}^{\frac{1}{2}}\nabla\varphi|^{p-2} \left(A_{sym}\nabla\varphi, \nabla v\right)\, dx. \end{aligned}\tag{2.9}$$

Here, the right-hand side of (2.9) is continuous in $v \in W_0^{1,p}(\Omega)$ and, therefore, represents an element of $W^{-1,q}(\Omega)$ because

$$\begin{aligned}&\int_\Omega |A_{sym}^{\frac{1}{2}}\nabla\varphi|^{p-2} \left(A_{sym}\nabla\varphi, \nabla v\right)\, dx \\ &\qquad \leq \left(\int_\Omega |A_{sym}^{\frac{1}{2}}\nabla\varphi|^p\, dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |A_{sym}^{\frac{1}{2}}\nabla v|^p\, dx\right)^{\frac{1}{p}} \\ &\leq \|\xi_2\|^p_{L^\infty(\Omega)} \|\nabla\varphi\|^{p-1}_{L^p(\Omega)^N} \|\nabla v\|_{L^p(\Omega)^N} = \|\xi_2\|^p_{L^\infty(\Omega)} \|\varphi\|^{p-1}_{W_0^{1,p}(\Omega)} \|v\|_{W_0^{1,p}(\Omega)}\end{aligned}$$

(we apply here Hölder's inequality and use the estimate $|A_{sym}^{\frac{1}{2}}\nabla\varphi|^p \le \xi_2^p|\nabla\varphi|^p$ coming from the condition $A_{sym}\in\mathfrak{A}_{ad}$). Hence, operator $\Delta_p(A_{sym},\cdot): W_0^{1,p}(\Omega)\to W^{-1,q}(\Omega)$ is bounded. The coercivity of $\Delta_p(A_{sym},\cdot)$ we get immediately, since

$$\langle \Delta_p(A_{sym},y),y\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \ge \alpha^p\|y\|^p_{W_0^{1,p}(\Omega)}.$$

As for the proof of strict monotonicity and semicontinuity of operator $\Delta_p(A_{sym},\cdot)$, we refer for the details to [8, 14]).

In what follows, we associate with $A_{skew}\in L^q(\Omega;\mathbb{S}^N_{skew})$ the bilinear skew-symmetric form

$$\Phi(y,v)=\int_\Omega \big(\nabla v, A_{skew}\nabla y\big)_{\mathbb{R}^N}\,dx,\quad \forall\, y,v\in W_0^{1,p}(\Omega). \tag{2.10}$$

It is easy to see, that the form $\Phi(y,v)$ is unbounded on $W_0^{1,p}(\Omega)$, in general.

However, if we temporary assume that $A_{skew}\in L^\infty(\Omega;\mathbb{S}^N_{skew})$, then the bilinear form $\Phi(\cdot,\cdot)$ becomes bounded on $W_0^{1,p}(\Omega)$. In order to deal with the case $A_{skew}\notin L^\infty(\Omega;\mathbb{S}^N_{skew})$, we notice that the value $\Phi(y,\varphi)$ is always finite provided $y\in W_0^{1,p}(\Omega)$ and $\varphi\in C_0^\infty(\Omega)$. Indeed,

$$\begin{aligned}|\Phi(y,\varphi)| &:= \left|\int_\Omega \big(\nabla\varphi, A_{skew}\nabla y\big)_{\mathbb{R}^N}\,dx\right| \le \|\varphi\|_{C^1(\overline{\Omega})}\left(\int_\Omega |A_{skew}\nabla y|_{\mathbb{R}^N}\,dx\right)\\ &\le \|\varphi\|_{C^1(\overline{\Omega})}\left(\int_\Omega \big\|A_{skew}\big\|^q\,dx\right)^{1/q}\left(\int_\Omega |\nabla y|^p_{\mathbb{R}^N}\,dx\right)^{1/p}\\ &\le \|\varphi\|_{C^1(\overline{\Omega})}\|A_{skew}\|_{L^q(\Omega;\mathbb{S}^N_{skew})}\|y\|_{W_0^{1,p}(\Omega)}<+\infty.\end{aligned}$$

Hence, if $A_{skew}\in L^q\big(\Omega;\mathbb{S}^N_{skew}\big)$ then the integral $\displaystyle\int_\Omega \big(\nabla\varphi, A_{skew}\nabla y\big)_{\mathbb{R}^N}\,dx$ is well defined for every $y\in W_0^{1,p}(\Omega)$ and $\varphi\in C_0^\infty(\Omega)$.

3. Setting of the Optimal Control Problem

Let $p\ge 1$ be a given exponent and let $f:\Omega\to\mathbb{R}^N$ be a vector-valued function such that $f\in L^q(\Omega;\mathbb{R}^N)$. The optimal control problem (1.1)–(1.3) we consider in this paper is to minimize the discrepancy (tracking error) between a given distribution $y_d\in L^p(\Omega)$ and a solution y of the Dirichlet boundary value problem (1.2)–(1.3) by choosing an appropriate matrix-valued control $A_{sym}(x)\in\mathfrak{A}_{ad}$.

Definition 3.1. We say that a function $y=y(A_{sym},A_{skew},f)$ is a weak solution (in the sense of Minty) to boundary value problem (1.2),(2.3),(2.4) for a fixed control $A_{sym}\in\mathfrak{A}_{ad}$, given matrix $A_{skew}\in L^q(\Omega;\mathbb{S}^N_{skew})$ and function $f\in L^q(\Omega)$ if $y\in W_0^{1,p}(\Omega)$ and

$$\begin{aligned}&\int_\Omega |A_{sym}^{1/2}\nabla\varphi|^{p-2}_{\mathbb{R}^N}(A_{sym}\nabla\varphi,\nabla\varphi-\nabla y)_{\mathbb{R}^N}\,dx\\ &+\int_\Omega (A_{skew}\nabla\varphi,\nabla\varphi-\nabla y)_{\mathbb{R}^N}\,dx \ge \int_\Omega (f,\nabla\varphi-\nabla y)_{\mathbb{R}^N}\,dx\quad \forall\,\varphi\in C_0^\infty(\Omega).\end{aligned} \tag{3.1}$$

We note that by the initial assumptions and Hölder's inequality, this definition makes a sense because $(A_{skew}\nabla y) \in L^1(\Omega;\mathbb{R}^N)$ for each $y \in W_0^{1,p}(\Omega)$.

It is worth to notice that the original boundary value problem (1.2)–(1.3) is ill-possed, in general. In view of the fact that the skew-symmetric form (2.10) can be unbounded on $W_0^{1,p}(\Omega)$, the existence of a weak solution to (1.2),(2.3),(2.4) for fixed $A_{sym} \in \mathfrak{A}_{ad}$, $A_{skew} \in L^q(\Omega;\mathbb{S}^N_{skew})$ and $f \in L^q(\Omega;\mathbb{R}^N)$ seems to be an open question.

On the other hand, the Minty inequality (3.1) does not allow us to derive a reasonable a priory estimate in $\|\cdot\|_{W_0^{1,p}(\Omega)}$-norm for the weak Minty solutions. Thus, in general, the mapping $A_{sym} \mapsto y(A_{sym}, A_{skew}, f)$ can be unbounded.

Taking these observations into account, we restrict our analysis to the following set of admissible solutions for the original optimal control problem. Namely, we indicate the set

$$\Xi = \left\{ (A_{sym}, y) \;\middle|\; A_{sym} \in \mathfrak{A}_{ad},\; y \in W_0^{1,p}(\Omega),\; (A_{sym}, y) \text{ are related by (1.2)} \right\}. \tag{3.2}$$

We adopt the following hypothesis, which is mainly motivated by the previous reasonings.

Hypothesis A. *The set of admissible solutions Ξ is nonempty.*

We say that a pair $(A^0_{sym}, y^0) \in L^\infty(\Omega;\mathbb{S}^N_{sym}) \times W_0^{1,p}(\Omega)$ is a weak optimal solution to the problem (1.2),(2.3),(2.4) on the set Ξ, if

$$(A^0_{sym}, y^0) \in \Xi \quad\text{and}\quad I(A^0_{sym}, y^0) = \inf_{(A_{sym},y)\in\Xi} I(A, y). \tag{3.3}$$

For our further analysis, we make use of the following results. We begin with the following property.

Lemma 3.1. *Let $\left\{(A_{sym_k}, y_k) \in \Xi\right\}_{k\in\mathbb{N}}$ be a sequence such that $A^{\frac{1}{2}}_{sym_k} \overset{*}{\rightharpoonup} A^{\frac{1}{2}}_{sym}$ in $BV(\Omega;\mathbb{S}^N_{sym})$ and $y_k \rightharpoonup y$ in $W_0^{1,p}(\Omega)$. Then*

$$\begin{aligned}\lim_{k\to\infty} &\int_\Omega |\left(\nabla\varphi, A_{sym_k}\nabla\varphi\right)|^{\frac{p-2}{2}} \left(\nabla y_k, A_{sym_k}\nabla\varphi\right)_{\mathbb{R}^N}\,dx \\ &= \int_\Omega |\left(\nabla\varphi, A_{sym}\nabla\varphi\right)|^{\frac{p-2}{2}} \left(\nabla y, A_{sym}\nabla\varphi\right)_{\mathbb{R}^N}\,dx, \quad \forall\varphi \in C_0^\infty(\Omega).\end{aligned} \tag{3.4}$$

Proof. Since $A^{\frac{1}{2}}_{sym_k} \to A^{\frac{1}{2}}_{sym}$ in $L^1(\Omega;\mathbb{S}^N)$ and the sequence $\{A^{\frac{1}{2}}_{sym_k}\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega;\mathbb{S}^N)$, by Lebesgue's Theorem we get that $A^{\frac{1}{2}}_{sym_k} \to A^{\frac{1}{2}}_{sym}$ strongly in $L^r(\Omega;\mathbb{S}^N)$ for every $1 \le r < +\infty$. Hence, $A^{\frac{1}{2}}_{sym_k}\nabla\varphi \to A^{\frac{1}{2}}_{sym}\nabla\varphi$ strongly in $L^p(\Omega)^N$ for every $\varphi \in C_0^\infty(\Omega)$. Therefore, for all $\varphi \in C_0^\infty(\Omega)$ we have

$$|A^{\frac{1}{2}}_{sym_k}\nabla\varphi|^{p-2}A^{\frac{1}{2}}_{sym_k}\nabla\varphi \to |A^{\frac{1}{2}}_{sym}\nabla\varphi|^{p-2}A^{\frac{1}{2}}_{sym}\nabla\varphi \text{ in } L^q(\Omega)^N. \tag{3.5}$$

Moreover, since $A^{\frac{1}{2}}_{sym_k}\nabla\psi \to A^{\frac{1}{2}}_{sym}\nabla\psi$ strongly in $L^q(\Omega)^N$ for every $\psi \in C_0^\infty(\Omega)$ and $\nabla y_k \rightharpoonup \nabla y$ in $L^p(\Omega)^N$, it follows that

$$\int_\Omega \left(A^{\frac{1}{2}}_{sym_k}\nabla y_k, \nabla\psi \right) dx = \int_\Omega \left(\nabla y_k, A^{\frac{1}{2}}_{sym_k}\nabla\psi \right) dx$$
$$\to \int_\Omega \left(\nabla y, A^{\frac{1}{2}}_{sym}\nabla\psi \right) dx = \int_\Omega \left(A^{\frac{1}{2}}_{sym}\nabla y, \nabla\psi \right) dx, \quad \forall \psi \in C_0^\infty(\Omega), \tag{3.6}$$

as a product of weakly and strongly convergent sequences in $L^p(\Omega)^N$ and $L^q(\Omega)^N$, respectively. Using the fact that

$$\sup_{k\in\mathbb{N}} \|A^{\frac{1}{2}}_{sym_k}\nabla y_k\|_{L^p(\Omega)^N} \le \|\xi_2\|_{L^\infty(\Omega)} \sup_{k\in\mathbb{N}} \|\nabla y_k\|_{L^p(\Omega)^N} < +\infty,$$

we finally get from (3.6)

$$A^{\frac{1}{2}}_{sym_k}\nabla y_k \rightharpoonup A^{\frac{1}{2}}_{sym}\nabla y \quad \text{in} \quad L^p(\Omega; \mathbb{R}^N). \tag{3.7}$$

Thus, to complete the proof it remains to note that

$$\int_\Omega | \left(\nabla\varphi, A_{sym_k}\nabla\varphi \right) |^{\frac{p-2}{2}} \left(\nabla y_k, A_{sym_k}\nabla\varphi \right) dx$$
$$= \int_\Omega \left(|A^{\frac{1}{2}}_{sym_k}\nabla\varphi|^{p-2} A^{\frac{1}{2}}_{sym_k}\nabla\varphi, A^{\frac{1}{2}}_{sym_k}\nabla y_k \right) dx$$

and apply properties (3.5) and (3.7). □

The following result deals with closedness of the set of admissible solutions Ξ with respect to a suitable topology.

Lemma 3.2. *Suppose Hypothesis A holds. Let* $\left\{(A_{sym_k}, y_k)\right\}_{k\in\mathbb{N}} \subset \Xi$ *be a sequence of admissible pair such that*

$$A^{\frac{1}{2}}_{sym_k} \stackrel{*}{\rightharpoonup} A^{\frac{1}{2}}_{sym_0} \quad in \quad BV(\Omega; \mathbb{S}^N_{sym}), \tag{3.8}$$
$$y_k \rightharpoonup y \quad in \quad W_0^{1,p}(\Omega), \tag{3.9}$$

Then $A_{sym_0} \in \mathfrak{A}_{ad}$, $(A_{sym_0}, y) \in \Xi$.

Proof. Since $A_{sym_k} \in \mathfrak{A}_{ad}$ for every $k \in \mathbb{N}$ then, in view of Remark 2.2, we immediately get $A_{sym_0} \in \mathfrak{A}_{ad}$. Moreover, it follows that the Minty inequality

$$\int_\Omega |A^{1/2}_{sym_k}\nabla\varphi|^{p-2}_{\mathbb{R}^N} (A_{sym_k}\nabla\varphi, \nabla\varphi - \nabla y_k)_{\mathbb{R}^N}\, dx$$
$$+ \int_\Omega (A_{skew}\nabla\varphi, \nabla\varphi - \nabla y_k)_{\mathbb{R}^N}\, dx \ge \int_\Omega (f, \nabla\varphi - \nabla y_k)_{\mathbb{R}^N}\, dx \tag{3.10}$$

holds true for all $\varphi \in C_0^\infty(\Omega)$. In order to pass to the limit in (3.10), it is enough to use Lemma 3.1. Taking this fact into account as well as weak convergence $\nabla y_k \rightharpoonup \nabla y$ in $L^p(\Omega;\mathbb{R}^N)$ we have

$$\int_\Omega |A_{sym_0}^{1/2}\nabla\varphi|_{\mathbb{R}^N}^{p-2}(A_{sym_0}\nabla\varphi, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\,dx \\ + \int_\Omega (A_{skew}\nabla\varphi, \nabla\varphi-\nabla y)_{\mathbb{R}^N}\,dx \geq \int_\Omega (f, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\,dx.$$

Therefore $(A_{sym_0}, y) \in \Xi$. □

We are now in a position to establish the main result of this section.

Theorem 3.1. *Assume that Hypothesis A is valid. Then the optimal control problem* (1.1)–(1.3) *admits at least one solution for all distributions* $A_{skew} \in L^q(\Omega;\mathbb{S}^N_{skew})$, $f \in L^q(\Omega;\mathbb{R}^N)$ *and* $y_d \in L^p(\Omega)$.

Proof. Since the original problem is regular and the cost functional for the given problem is bounded below on Ξ, it follows that there exists a minimizing sequence $\{(A_{sym_k}, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ such that

$$I(A_{sym_k}, y_k) \xrightarrow[k\to\infty]{} I_{\min} \equiv \inf_{(A_{sym},y)\in\Xi} I(A_{sym}, y) \geq 0.$$

Hence, $\sup_{k\in\mathbb{N}} I(A_{sym_k}, y_k) \leq C$, where the constant C is independent of k. Since $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, we have

$$\begin{aligned}\sup_{k\in\mathbb{N}} \|y_k\|^p_{W_0^{1,p}(\Omega)} &= \int_\Omega \left(y_k^p + |\nabla y_k|^p_{\mathbb{R}^N}\right) dx \\ &\leq \max\left\{1, (\varepsilon\alpha^p)^{-1}\right\} \int_\Omega \left(y_k^p + \varepsilon |A^{\frac{1}{2}}_{sym_k}\nabla y_k|^p_{\mathbb{R}^N}\right) dx \\ &\leq 2^{p-1}\max\left\{1, (\varepsilon\alpha^p)^{-1}\right\}\left(\sup_{k\in\mathbb{N}} I(A_k, y_k) + \|y_d\|^p_{L^p(\Omega)}\right) \\ &\leq 2^{p-1}\max\left\{1, (\varepsilon\alpha^p)^{-1}\right\}\left(C + \|y_d\|^p_{L^p(\Omega)}\right).\end{aligned} \tag{3.11}$$

Therefore, passing to a subsequence if necessary and taking into account Remark 2.2, we may assume the existence of a pair $(A_{sym_0}, y_0) \in \mathfrak{A}_{ad} \times W_0^{1,p}(\Omega)$ such that

$$A^{\frac{1}{2}}_{sym_k} \overset{*}{\rightharpoonup} A^{\frac{1}{2}}_{sym_0} \text{ in } BV(\Omega;\mathbb{S}^N_{sym}), \tag{3.12}$$

$$y_k \rightharpoonup y_0 \text{ in } L^p(\Omega), \tag{3.13}$$

$$\nabla y_k \rightharpoonup \nabla y_0 \text{ in } L^p(\Omega, \mathbb{R}^N\,dx), \tag{3.14}$$

$$I(A_{sym_0}, y_0) < +\infty. \tag{3.15}$$

In remains to show that (A_0, y_0) is an optimal pair. Using conditions (3.13)–(3.15) and the property of lower semicontinuity of the norms $\|\cdot\|_{L^p(\Omega;\mathbb{R}^N)}$ and

$\|\cdot\|_{L^p(\Omega)}$ with respect to the weak topologies of $L^p(\Omega;\mathbb{R}^N\,dx)$ and $L^p(\Omega)$, as well as properties (2.2), we get

$$\liminf_{k\to\infty}\|y_k-y_d\|^p_{L^p(\Omega)}\ge\|y_0-y_d\|^p_{L^p(\Omega)}\,,$$
$$\liminf_{k\to\infty}\int_\Omega\left(\nabla y_k,A_{sym_k}\nabla y_k\right)^{p/2}_{\mathbb{R}^N}\,dx\ge\int_\Omega\left(\nabla y_0,A_{sym_0}\nabla y_0\right)^{p/2}_{\mathbb{R}^N}\,dx.$$

Indeed, to prove last relation, it is enough to show weak convergence of the sequence $\{A^{\frac{1}{2}}_{sym_k}\nabla y_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^N)$ to element $A^{\frac{1}{2}}_{sym_0}\nabla y_0\in L^p(\Omega;\mathbb{R}^N)$ and use lower semi-continuity property of the norm in $L^p(\Omega;\mathbb{R}^N)$ as follows

$$\begin{aligned}\liminf_{k\to\infty}\int_\Omega\left(\nabla y_k,A_{sym_k}\nabla y_k\right)^{p/2}_{\mathbb{R}^N}\,dx&=\liminf_{k\to\infty}\int_\Omega|A^{\frac{1}{2}}_{sym_k}\nabla y_k|^p_{\mathbb{R}^\mathbb{N}}dx\\&=\liminf_{k\to\infty}\|A^{\frac{1}{2}}_{sym_k}\nabla y_k\|^p_{L^p(\Omega;\mathbb{R}^N)}\ge\|A^{\frac{1}{2}}_{sym_0}\nabla y_0\|^p_{L^p(\Omega;\mathbb{R}^N)}\\&=\int_\Omega\left(\nabla y_0,A_{sym_0}\nabla y_0\right)^{p/2}_{\mathbb{R}^N}\,dx.\end{aligned}$$

Mentioned weak convergence $A^{\frac{1}{2}}_{sym_k}\nabla y_k\rightharpoonup A^{\frac{1}{2}}_{sym_0}\nabla y_0$ in $L^p(\Omega;\mathbb{R}^N)$ comes from the argumentation of Lemma 3.1, see (3.6)–(3.7).

Thus,

$$\begin{aligned}I(A_0,y_0)&\ge\inf_{(A,y)\in\Xi}I(A,y)=\lim_{k\to\infty}I(A_k,y_k)\ge\liminf_{k\to\infty}I(A_k,y_k)\\&\ge\|y_0-y_d\|^p_{L^p(\Omega)}+\varepsilon\int_\Omega(\nabla y_0,A_0^{sym}\nabla y_0)^{p/2}_{\mathbb{R}^N}\,dx=I(A_0,y_0),\end{aligned}$$

and, hence, the pair (A_0,y_0) is optimal for problem (1.1)–(1.3). The proof is complete. □

In what follows we prove that Hypothesis A takes place, i.e. there exists at least one admissible pair $(A_{sym},y)\in\Xi$. With that in mind we consider the sequence $A_{skew_k}\in L^\infty(\Omega;\mathbb{S}^N_{skew})$, such that $A_{skew_k}\to A_{skew}$ strongly in $L^q(\Omega;\mathbb{S}^N_{skew})$ as $k\to\infty$ (existence of such sequence is a well-known fact of functional analysis as well as methods of its construction). Now we fix any matrix $A_{sym}\in\mathfrak{A}_{ad}$, and with it and matrix A_{skew} we associate the corresponding sequence of boundary value problems

$$-\mathrm{div}\left(|(A_{sym}\nabla y,\nabla y)|^{\frac{p-2}{2}}A_{sym}\nabla y\right)-\mathrm{div}\left(A_{skew_k}\nabla y\right)=-\mathrm{div}\,f,\tag{3.16}$$
$$y\in W_0^{1,p}(\Omega).\tag{3.17}$$

It is easy to see that the differential operator in the left-hand of (3.16) possesses strict monotonicity, coercivity and semi-continuity properties on $W_0^{1,p}(\Omega)$, and, hence (see, for instance, [4,5]), boundary value problem (3.16)–(3.17) has a unique weak solution, i.e. for each $k \in \mathbb{N}$ there exists a unique $y_k \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |A_{sym}^{1/2} \nabla y_k|_{\mathbb{R}^N}^{p-2} (A_{sym} \nabla y_k, \nabla \varphi)_{\mathbb{R}^N}\, dx + \int_\Omega (A_{skew_k} \nabla y_k, \nabla \varphi)_{\mathbb{R}^N}\, dx = \int_\Omega (f, \nabla \varphi)_{\mathbb{R}^N}\, dx \;\forall\, \varphi \in C_0^\infty(\Omega). \quad (3.18)$$

Remark 3.1. In fact, as far as $A_{skew_k} \in L^\infty(\Omega; \mathbb{S}^N_{skew})$, the bilinear form

$$\int_\Omega (A_{skew_k} \nabla y, \nabla v)\, dx$$

is bounded on $W_0^{1,p}(\Omega)$ and, hence, the integral identity (3.18) takes place if and only if y_k is a weak Minty solution of (3.16)–(3.17) (see Definition 3.1). Indeed, following S.E. Pastukhova [9] this fact can be proved, using argumentation called the Minty trick. First, starting from Minty inequality

$$\int_\Omega |A_{sym}^{1/2} \nabla \varphi|_{\mathbb{R}^N}^{p-2} (A_{sym} \nabla \varphi, \nabla \varphi - \nabla y_k)_{\mathbb{R}^N}\, dx + \int_\Omega (A_{skew_k} \nabla \varphi, \nabla \varphi - \nabla y_k)_{\mathbb{R}^N}\, dx \geq \int_\Omega (f, \nabla \varphi - \nabla y_k)_{\mathbb{R}^N}\, dx, \quad (3.19)$$

we obtain the integral identity (3.18). Indeed, since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, as test functions in (3.19) we can take $\varphi = y_k \pm tv$, where $t \in \mathbb{R}^N$, $v \in W_0^{1,p}(\Omega)$. After passing to the limit, taking into account semi-continuity of $\Delta_p(A_{sym}, \cdot)$, we have

$$\pm \left(\int_\Omega |A_{sym}^{1/2} \nabla y|_{\mathbb{R}^N}^{p-2} (A_{sym} \nabla y_k, \nabla v)_{\mathbb{R}^N}\, dx + \int_\Omega (A_{skew_k} \nabla y_k, \nabla v)_{\mathbb{R}^N}\, dx \right) \geq \pm \int_\Omega (f, \nabla v)_{\mathbb{R}^N}\, dx, \;\forall\, v \in W_0^{1,p}(\Omega),$$

which yields (3.18). Now, the other way round, we start from the identity (3.18). Using it and monotonicity property of considered differential operator, for any $v \in C_0^\infty(\Omega)$ we have

$$0 \leq \int_\Omega \left(|A_{sym}^{\frac{1}{2}} \nabla v|^{p-2} A_{sym} \nabla v - |A_{sym}^{\frac{1}{2}} \nabla y_k|^{p-2} A_{sym} \nabla y_k, \nabla \varphi \right)_{\mathbb{R}^N} dx + \int_\Omega (A_{skew_k} (\nabla v - \nabla y_k), \nabla \varphi)_{\mathbb{R}^N}\, dx =$$

$$\int_\Omega \left(|A_{sym}^{\frac{1}{2}} \nabla v|^{p-2} A_{sym} \nabla v, \nabla \varphi \right)_{\mathbb{R}^N} dx + \int_\Omega (A_{skew_k} \nabla v, \nabla \varphi)_{\mathbb{R}^N}\, dx - \int_\Omega \left(|A_{sym}^{\frac{1}{2}} \nabla y_k|^{p-2} A_{sym} \nabla y_k, \nabla \varphi \right)_{\mathbb{R}^N} dx - \int_\Omega (A_{skew_k} \nabla y_k, \nabla \varphi)_{\mathbb{R}^N}\, dx$$

$$= \langle \Delta_p(A_{sym}, v), \varphi \rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} - \int_\Omega (f, \nabla \varphi)_{\mathbb{R}^N} dx, \text{ for } \varphi = v - y_k.$$

We have thus established the desired relation

$$\langle \Delta_p(A_{sym}, v), v - y_k \rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} + \int_\Omega (A_{skew_k} \nabla v, \nabla v - \nabla y_k)_{\mathbb{R}^N}\, dx \\ \ge \int_\Omega (f, \nabla v - \nabla y_k)_{\mathbb{R}^N} dx, \ \forall\, v \in C_0^\infty(\Omega).$$

Together with integral identity (3.18) for each element y_k we get the energy equality

$$\int_\Omega |A_{sym}^{\frac{1}{2}} \nabla y_k|^p_{\mathbb{R}^N}\, dx + \int_\Omega (A_{skew_k} \nabla y_k, \nabla y_k)_{\mathbb{R}^N}\, dx = \int_\Omega (f, \nabla y_k)_{\mathbb{R}^N}\, dx. \quad (3.20)$$

However, taking into account the skew-symmetric property of A_{skew_k}, i.e.

$$(A_{skew_k} \nabla y, \nabla v)_{\mathbb{R}^N} = -(\nabla v, A_{skew_k} \nabla y)_{\mathbb{R}^N},$$

which yields $(A_{skew_k} \nabla y_k, \nabla y_k)_{\mathbb{R}^N} = 0$, we obtain the following a priory estimate: $\|y_k\|^{p-1}_{W_0^{1,p}(\Omega)} \le \alpha^{-p} \|f\|^q_{L^q(\Omega;\mathbb{R}^N)}$, for all $k \in \mathbb{N}$. Therefore, there exists an element $y^* \in W_0^{1,p}(\Omega)$ such that up to a subsequence $y_k \rightharpoonup y^*$ in $W_0^{1,p}(\Omega)$. Next, we show that y^* is a weak Minty solution of the initial boundary value problem (1.2), corresponding to the fixed control matrix $A_{sym} \in \mathfrak{A}_{ad}$. It is enough to pass to the limit in (3.19) as $k \to \infty$, taking into account strong convergence $A_{skew_k} \to A_{skew}$ in $L^q(\Omega; \mathbb{S}^N_{skew})$ and weak convergence $\nabla y_k \rightharpoonup \nabla y^*$ in $L^p(\Omega; \mathbb{R}^N)$. Obviously, we get

$$\int_\Omega |A_{sym}^{1/2} \nabla \varphi|^{p-2}_{\mathbb{R}^N} (A_{sym} \nabla \varphi, \nabla \varphi - \nabla y^*)_{\mathbb{R}^N}\, dx \\ + \int_\Omega (A_{skew} \nabla \varphi, \nabla \varphi - \nabla y^*)_{\mathbb{R}^N}\, dx \ge \int_\Omega (f, \nabla \varphi - \nabla y^*)_{\mathbb{R}^N}\, dx,$$

and, therefore, $(A_{sym}, y^*) \in \Xi$.

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